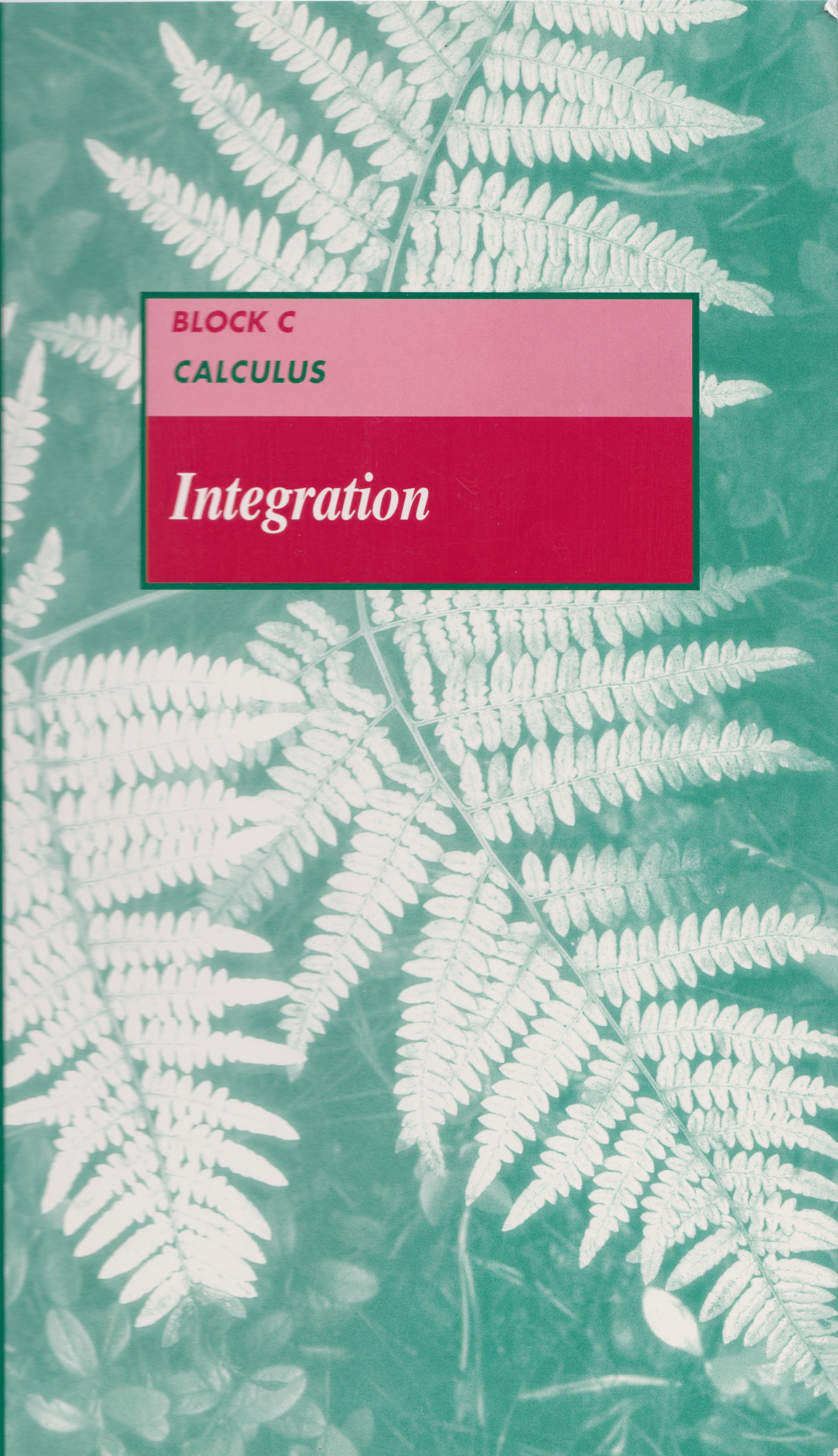
MS221 Chapter C2

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK C

CALCULUS

Integration

CHAPTER C2

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A second level interdisciplinary course

Exploring Mathematics

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CALCULUS

Integration

Prepared by the course team

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This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software program Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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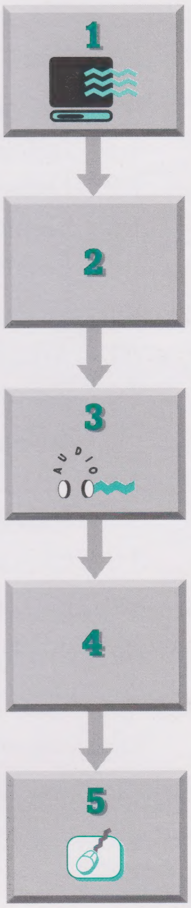

There are five sections to this chapter. They are intended to be studied consecutively in five study sessions.

Section 1 requires the use of a DVD player, and in Section 3 you will need access to an audio CD player. Section 5 requires the use of the computer together with Computer Book C.

The pattern of study for each session might be as follows.

Study session 1: Section 1.
Study session 2: Section 2.
Study session 3: Section 3.
Study session 4: Section 4.
Study session 5: Section 5.

The sections in this chapter are not of equal length or importance. You are encouraged to allocate a significant proportion of your study time to Sections 2 and 3, which lie at the core of the chapter. In terms of pages Section 1 is lengthy, but Subsection 1.4 (including the video band) will not be assessed. Also it is likely that you will be familar with much of the content of Section 1 (see below), which deals with basic ideas in integration.

Throughout the chapter you should concentrate your efforts on the various integration techniques introduced, and not spend an excessive amount of time on the more abstract aspects of the material.

In this chapter it is often convenient to refer to a general function as $f(x)$ rather than f. This is a common abuse of notation.

The material in this chapter is self-contained. However, it is likely that you already have some familiarity with the following topics:

- the indefinite integral and the definite integral and the associated notations $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$;
- finding the integrals of combinations of certain basic functions such as $\int (e^x + x^n)\,dx$ and $\int (\sin(2x) - \cos x)\,dx$.

The optional Video Band C(ii) 'Algebra Workout – Integration' could be viewed at any stage during your study of this chapter.

Introduction

The process of differentiation discussed in Chapter C1 enables us to determine the gradient of the graph of a smooth function at any point on it. Often, however, we are faced with the reverse problem, in which we *know* the gradient at each point of a graph and wish to determine a function that gives rise to the graph. This process of reversing differentiation is known as **integration**, and it is the topic we discuss in this chapter.

In general, if two functions F and f are such that $F' = f$, then F is said to be an *antiderivative* or *integral* of f. Sometimes we can find an integral of a given function f by recognising that f is one of the standard derivatives tabulated in the previous chapter; this enables us to write down an integral directly. On other occasions a process of 'guessing and checking' enables us to 'tease out' an integral of f by checking that its derived function is f.

A remarkable result links the antiderivatives of any *continuous* function f with areas that lie between the graph of f and the x-axis. These areas lead us to define numerical quantities known as *definite integrals* of f over intervals $[a, b]$. Such integrals can be evaluated by subtracting the value of any antiderivative of f at a from its value at b. A result known as the *Fundamental Theorem of Calculus* ensures that any continuous function has an antiderivative.

Informally, a continuous function is one whose graph can be drawn without lifting pen from paper. A rigorous definition of continuity is beyond the scope of this course.

Unlike differentiation, where we have rules for differentiating all products and composites of standard functions, no such general rules are available for integration. However two techniques known as *integration by parts* and *integration by substitution* enable us to integrate certain functions by 'undoing' the Product and Composite Rules of differentiation, respectively.

With experience these techniques become easier to apply, but there is no guarantee of success. Some computer integration packages are remarkably good at integrating certain functions automatically but, as you will see, they can sometimes give misleading or confusing answers. They should therefore be used with caution, to back up your theoretical understanding of integration, not to replace it.

Section 1 introduces the *indefinite* and *definite* integrals of a function f, and describes how the *Sum* and *Constant Multiple Rules* can sometimes be used to evaluate them. It also links the integrals of f to areas under the graph of f and discusses the *Fundamental Theorem of Calculus.*

Section 2 introduces a technique known as *integration by parts*, which arises from the Product Rule of differentiation.

Section 3 introduces a technique known as *integration by substitution*, which arises from the Composite Rule of differentiation.

Section 4 shows how the area under the graph of a function can be rotated about the x-axis to produce *a solid of revolution*, such as a cylinder, a cone, or a sphere. You will see how to calculate the volume of such a solid by using a definite integral.

Section 5 explores integration on the computer in the light of the experience provided by earlier sections.

1 Integration

To study Subsection 1.4, you will need a DVD player and DVD00115.

1.1 Indefinite integral

As mentioned in the introduction, this chapter concentrates on a process known as *integration* (or *antidifferentiation*) that reverses the process of differentiation. If F and f are functions that satisfy the equation

$$F'(x) = f(x) \quad \text{for all } x \in I,$$

where I is an interval in the domains of F and f, then F is called an **antiderivative**, or **integral, of f over I**. For example,

$F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$ over $\mathbb{R}$; and

$F(x) = \ln x$ is an antiderivative of $f(x) = 1/x$ over $(0, \infty)$.

Antiderivatives are not unique, for if $F(x)$ has derivative $f(x)$ over some interval I, then $F(x) + 3$ also has derivative $f(x)$. Indeed, if c is *any* constant, then $F(x) + c$ has derivative $f(x)$. So the expression $F(x) + c$ describes the entire family of antiderivatives of f over the specified interval I, one for each value of c. For this reason we refer to $F(x) + c$ as the **indefinite integral of $f(x)$ over I**; this indefinite integral is denoted by

$$\int f(x)dx \quad (x \in I).$$

The interval I over which an integral is defined is often omitted, particularly when it coincides with the domain of f.

So, if F is any antiderivative of f over I, then we can write

$$\int f(x)dx = F(x) + c \quad (x \in I).$$

Here, the symbol $\int$ is known as the **integral sign**, and the dx indicates that we are integrating with respect to the variable x, thereby undoing the effect of differentiation with respect to the variable x. The function $f(x)$ is called the **integrand** and we say that $F(x) + c$ is obtained by **integrating** $f(x)$. The constant c is called an **arbitrary constant**, or the **constant of integration**.

A function may not have any antiderivatives, in which case it cannot be integrated. Fortunately, the functions we shall consider are *continuous* on all intervals in their domains, and for reasons that are explained later in the section, such functions can always be integrated.

One of the simplest ways to find the indefinite integral of a function f is to recognise that $f(x)$ is a standard derivative. We know, for example, that $5x^4$ is the derivative of x^5, so this enables us to write

$$\int 5x^4\, dx = x^5 + c,$$

where c is an arbitrary constant. Similarly,

$$\int \cos x\, dx = \sin x + c.$$

The table below, in which $a \neq 0$, lists other derivatives from Chapter C1 that are often used to determine indefinite integrals.

Table 1.1

Function $F(x)$	Derivative $F'(x)$
c	0
x^n	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
e^{ax}	ae^{ax}
$\ln\lvert x\rvert$	$\dfrac{1}{x} \quad (x \neq 0)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$
$\arctan x$	$\dfrac{1}{1+x^2}$

In the first entry c is a real number, and in the second n is a non-zero real number.

In Activity 2.9(b) of Chapter C1 you found that the derivative of the function $f(x) = \ln\lvert x\rvert$ is

$$f'(x) = 1/x \quad (x \neq 0).$$

Although the table expresses the derivatives in terms of the variable x, any other variable can be used to carry out an integration. For example, the third table entry (with $a = 2$) enables us to deduce that

$$\int 2\cos(2t)\,dt = \sin(2t) + c,$$

where the integration has been carried out with respect to the variable t, and c is an arbitrary constant.

Activity 1.1 Using the table

Use Table 1.1 to find the following indefinite integrals.

(a) $\displaystyle\int \frac{1}{1+u^2}\,du$ (b) $\displaystyle\int 3\cos(3x)\,dx$ (c) $\displaystyle\int 7e^{7t}\,dt$

Solutions are given on page 50.

The interval over which a function f is integrated must avoid any points where f is undefined. The function $f(x) = \sec^2 x$, for example, is undefined at the points $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$, so the chosen domain of its indefinite integral must avoid these points. Sometimes the choice of domain depends on what we want to do with the integral. If, for example, we are interested in the behaviour of the antiderivatives of $f(x) = \sec^2 x$ near 0, then we would choose a domain like the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ that includes 0, and then write

$$\int \sec^2 x\,dx = \tan x + c \quad \left(-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi\right).$$

We could choose a smaller interval such as $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$.

Activity 1.2 Specifying a domain

Use Table 1.1 to find the following indefinite integrals. In each case suggest a domain that includes points close to $-\frac{1}{2}$.

(a) $\displaystyle\int \frac{1}{x}\,dx$ (b) $\displaystyle\int \sec x \tan x\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

Solutions are given on page 50.

When making general statements about indefinite integrals, the presence of a variable, such as x, can obscure the fact that the integration process is concerned with functions. For this reason we sometimes drop the variable from the indefinite integral and write it more concisely in the form $\int f$. This purely functional way of writing indefinite integrals is particularly useful when stating rules of integration, such as the following *Sum* and *Constant Multiple* Rules, which apply to continuous functions.

Notice, however, that writing $\int f(x)$ without the dx is incorrect. Whenever the integrand is expressed in terms of variables, you must indicate the variable with respect to which you are integrating.

Sum and Constant Multiple Rules (for indefinite integrals)

$$\int (f+g) = \int f + \int g \quad \text{(Sum Rule);}$$

$$\int (kf) = k\int f \quad \text{(Constant Multiple Rule),}$$

where k is a non-zero real number.

In *this* Constant Multiple Rule, k must be non-zero since $\int(0\times f) = \int 0 = c$, an arbitrary constant, whereas $0\int f = 0$.

When using either of these rules, the integrands involved must be continuous on a common interval.

To see that these rules are correct, let F and G be antiderivatives of f and g, respectively. Then $(F+G)' = F' + G' = f + g$, so $F+G$ is an antiderivative of $f+g$. The Sum Rule follows from this fact by observing that the sum of two arbitrary constants $c_1 + c_2$ can take *any* real value, and is therefore itself just an arbitrary constant c. Hence

$$\begin{aligned}\int f(x)\,dx + \int g(x)\,dx &= (F(x)+c_1) + (G(x)+c_2)\\ &= F(x) + G(x) + c = \int (f(x)+g(x))\,dx,\end{aligned}$$

as required.

Similarly $(kF)' = kF' = kf$, so kF is an antiderivative of kf. The Constant Multiple Rule follows from this fact by observing that the multiple kc_1 can take any real value, and is therefore itself an arbitrary constant c. Hence

The multiple kc_1 can take any value because k is non-zero.

$$k\int f(x)\,dx = k(F(x)+c_1) = kF(x) + c = \int kf(x)\,dx,$$

as required.

Having discussed the Constant Multiple Rule, we are now in a position to integrate a function like $f(x) = \cos(5x)$. Since

$$\frac{d}{dx}(\sin(5x)) = 5\cos(5x),$$

it follows from the Constant Multiple Rule that

$$\int \cos(5x)\,dx = \tfrac{1}{5}\int 5\cos(5x)\,dx = \tfrac{1}{5}\sin(5x) + c,$$

where c is an arbitrary constant.

The fraction $\frac{1}{5}$ can be 'absorbed' into the arbitrary constant c, allowing us to write $\cdots + c$ rather than $\cdots + \frac{1}{5}c$.

More generally, if $a \neq 0$, then a similar argument tells us that

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c.$$

In a similar way, we can integrate functions of the form $f(x) = e^{ax}$, $f(x) = \sin(ax)$ and $f(x) = \cos(ax)$. We can then use these, along with some of the other integrals derived from Table 1.1, to draw up the following table of *standard indefinite integrals.*

Table 1.2

Function $f(x)$	Integral $\int f(x)\,dx$
a	$ax + c$
x^n $(n \neq -1)$	$\dfrac{1}{n+1}x^{n+1} + c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\dfrac{1}{a}e^{ax} + c$
$\cos(ax)$	$\dfrac{1}{a}\sin(ax) + c$
$\sin(ax)$	$-\dfrac{1}{a}\cos(ax) + c$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax) + c$
$\operatorname{cosec}^2(ax)$	$-\dfrac{1}{a}\cot(ax) + c$
$\sec(ax)\tan(ax)$	$\dfrac{1}{a}\sec(ax) + c$
$\operatorname{cosec}(ax)\cot(ax)$	$-\dfrac{1}{a}\operatorname{cosec}(ax) + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\arccos x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$

In e^{ax} and the entries below it, the constant a is non-zero.

When you choose a domain for an indefinite integral it must be an interval that avoids points where the integrand is undefined. For example, for $f(x) = \sec^2 x$ you must avoid the points

$$\pm\tfrac{1}{2}\pi,\ \pm\tfrac{3}{2}\pi,\ \pm\tfrac{5}{2}\pi,\ \ldots,$$

and for $f(x) = 1/\sqrt{1-x^2}$ you must avoid points outside the interval $(-1, 1)$.

By applying the Sum and Constant Multiple Rules we can now integrate combinations of functions drawn from this table. For example,

$$\begin{aligned}\int (4\sin(2t) - 6t^2)\,dt &= 4\int \sin(2t)\,dt - 6\int t^2\,dt\\ &= 4(-\tfrac{1}{2}\cos(2t)) - 6(\tfrac{1}{3}t^3) + c\\ &= -2\cos(2t) - 2t^3 + c,\end{aligned}$$

where c is an arbitrary constant.

Both the constants of integration have been 'absorbed' into a single arbitrary constant c.

Table 1.2 is reproduced in your Handbook, and from now on you should feel free to draw on any of its entries as standard integrals. The following activity provides you with practice in applying the Sum and Constant Multiple Rules to these standard integrals.

Activity 1.3 Integration practice

In part (b) you will need to think about how to rearrange the integrand to give a sum of two standard integrals.
In part (e) you will need the identity

$$\cos(2x) = 2\cos^2 x - 1.$$

Find each of the following indefinite integrals.

(a) $\displaystyle\int (5x + x^3)\,dx$ (b) $\displaystyle\int \frac{1+\sqrt{x}}{x}\,dx$ (c) $\displaystyle\int (u + 2e^{7u})\,du$

(d) $\displaystyle\int (\sin(2t) - 3\cos(3t))\,dt$ (e) $\displaystyle\int \cos^2 x\,dx$

Solutions are given on page 50.

1.2 Definite integral

The indefinite integral of a function f specifies the family of functions that are antiderivatives of f. In this subsection we turn our attention to a related, but different, form of integral known as a *definite* integral. Rather than specifying a family of antiderivatives of f such an integral yields a *numerical* value associated with the area of a region between the graph of f and the x-axis. There is a close connection between these two apparently quite different types of integral.

By a *region* we mean a two-dimensional set with a boundary. In some courses, the word region has other meanings.

To see how this works, suppose that the function f is non-negative on an interval $[a, b]$; that is, $f(x) \geq 0$ for all x in $[a, b]$. Thus the graph of f never dips below the x-axis on this interval. Let $A(x)$ be the area of the region under the graph of f from a to x, where $x \leq b$. That is, $A(x)$ is the area of the region bounded by the x-axis, the curve $y = f(x)$ and the vertical lines through a and x on the x-axis, as illustrated in Figure 1.1. The function A is often called the *area-so-far function.*

The more general case is discussed in the next subsection.

A similar, but slightly less formal version of the following argument was given in MST121, Chapter C2.

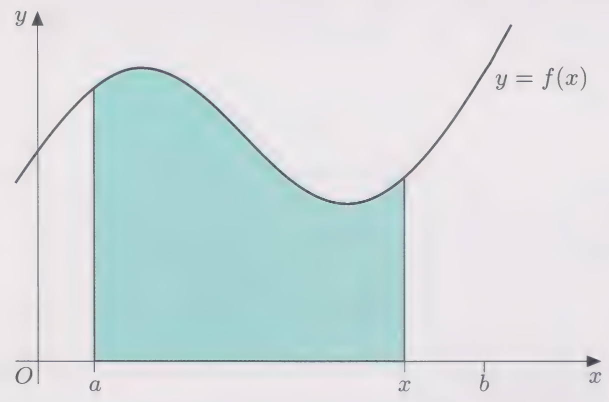

Figure 1.1 Area $A(x)$ is shaded

First, note that $A(a) = 0$, since if $x = a$, then the region with area $A(x)$ has zero width.

Now consider also the area $A(x + h)$, for which the right-hand bounding vertical line in Figure 1.1 is moved a small distance h to the right. Figure 1.2 shows the area of the region between the vertical lines at x and at $x + h$, which is equal to

The argument assumes that $h > 0$, but a similar argument applies for the case $h < 0$.

$$A(x + h) - A(x).$$

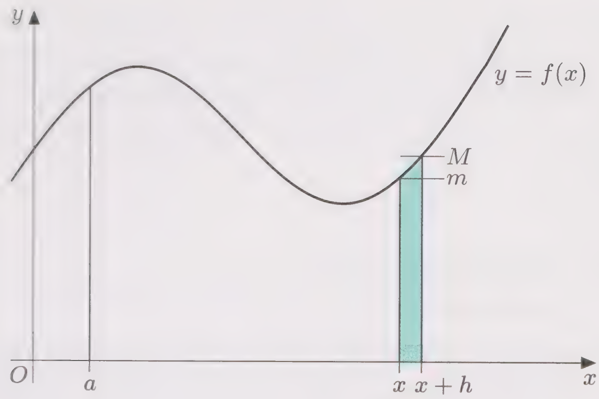

Figure 1.2 Area $A(x+h) - A(x)$ is shaded

In Figure 1.2, the function illustrated is increasing on $[x, x+h]$, so in this case $m = f(x)$ and $M = f(x+h)$.

Over the interval $[x, x+h]$, the values of the function f lie between some minimum value m and some maximum value M. The rectangle with height m and width h has area less than or equal to that beneath the graph of f on this interval, while the rectangle with height M and width h has area greater than or equal to that beneath the graph of f. These facts can be expressed by the inequalities

$$mh \leq A(x+h) - A(x) \leq Mh.$$

On dividing through by h, we obtain

$$m \leq \frac{A(x+h) - A(x)}{h} \leq M.$$

We now take the limit of each of these expressions as $h \to 0$. By the definition of the derivative, the limit of the middle term is

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x).$$

As $h \to 0$, the strip between x and $x+h$ becomes arbitrarily thin, and we would expect both m and M to tend to $f(x)$. This is true, provided that f is continuous on $[a, b]$. We obtain

Figure 1.2 indicates that this assertion is reasonable, but a proof would require a rigorous definition of continuity.

$$f(x) \leq A'(x) \leq f(x),$$

and hence

$$A'(x) = f(x).$$

By integration it follows that

$$A(x) = F(x) + c \quad (x \in [a, b]),$$

where F is *any* antiderivative of f, and c is an appropriate constant of integration. We can determine this constant by observing that when $x = a$ the area $A(a) = F(a) + c$ must be zero, so $c = -F(a)$. Hence

$$A(x) = F(x) - F(a).$$

In particular, putting $x = b$ we discover that

$$A(b) = F(b) - F(a).$$

Since $A(b)$ is the area under the graph of f from a to b, we have succeeded in establishing the following result.

Let f be a function that is continuous on an interval $[a, b]$. If f has the property that $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$, is equal to

$$F(b) - F(a),$$

where F is any antiderivative of f.

Differences of the form $F(b) - F(a)$ occur so frequently in integration that they are often expressed in a specially abbreviated form. We write

In $[F(x)]_a^b$, it is not required that $a < b$.

$$F(b) - F(a) = [F(x)]_a^b,$$

which is sometimes further abbreviated to $[F]_a^b$.

For example, in the case where $F(x) = \frac{1}{2}x^2$ we can evaluate

$$\left[\tfrac{1}{2}x^2\right]_{-1}^{2} = \tfrac{1}{2}(2)^2 - \tfrac{1}{2}(-1)^2 = 2 - \tfrac{1}{2} = \tfrac{3}{2}$$

and

$$\left[\tfrac{1}{2}x^2\right]_{2}^{0} = \tfrac{1}{2}(0)^2 - \tfrac{1}{2}(2)^2 = -2.$$

Now suppose that F and G are two antiderivatives of a function f, where f is no longer required to be non-negative. Since $G(x) = F(x) + k$ for some constant k, it follows that

$$G(b) - G(a) = (F(b) + k) - (F(a) + k) = F(b) - F(a).$$

This equation implies that the value of $[F(x)]_a^b$ is independent of the particular choice of antiderivative F, so it can be thought of as a quantity associated with f rather than F. We call this quantity the *definite integral* of f from a to b.

Definite integral of a function

Let f be a continuous function with antiderivative F over an interval I, and let $a, b \in I$.

The phrase 'from a to b' does not imply that $a < b$.

The **definite integral of f from a to b**, denoted by

$$\int_a^b f(x)\,dx \quad \text{or by} \quad \int_a^b f,$$

is defined to be

$$[F(x)]_a^b = F(b) - F(a).$$

The number a is called the **lower limit** of integration and the number b is called the **upper limit** of integration.

It is important to realise that the above definition applies to *any* continuous function f. In particular it does not require f to be non-negative nor does it require $a < b$. But having introduced the idea of a definite integral for an arbitrary continuous function we can now use it to restate our previous result about the area under the graph of a non-negative function f.

Area under a graph

Let f be a function that is continuous on an interval $[a, b]$.

If f has the property that $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$, is equal to

$$\int_a^b f(x)\,dx.$$

This area is referred to as the *area under the graph of $y = f(x)$ from $x = a$ to $x = b$.*

For example, if $f(x) = \cos x$, then we can use the antiderivative $F(x) = \sin x$ to calculate the area under the graph of $y = f(x)$ from $x = -1$ to $x = 1$; see Figure 1.3. It is given by the definite integral

$$\begin{aligned}\int_{-1}^{1} \cos x\,dx &= [\sin x]_{-1}^{1} \\ &= \sin(1) - \sin(-1) \\ &= 1.68 \text{ (to 2 d.p.).}\end{aligned}$$

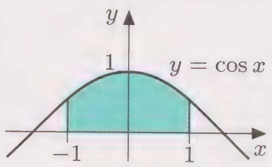

Figure 1.3 Graph of $y = \cos x$

This calculation provides the area because the function f is continuous, and its graph never dips below the x-axis throughout the interval $[-1, 1]$. It is therefore meaningful to talk about the area under the graph of f from -1 to 1. The following activity invites you to consider some similar cases where it is meaningful to talk about the area under the graph of f.

In fact, $f(x) = \cos x \geq 0$ for all $x \in [-\pi/2, \pi/2]$.

Activity 1.4 Finding areas

(a) Sketch the area under the graph of $y = e^{-x/2}$ from $x = 0$ to $x = 4$. Calculate the area to two decimal places.

(b) Sketch, and then calculate, the area under the graph of $y = \cos(3x)$ from $x = 0$ to $x = \pi/6$.

(c) Sketch, and then calculate, the area bounded by the parabola $y = 1 - x^2$ and the x-axis.

(d) Sketch the smaller area bounded by the circle $x^2 + y^2 = 1$ and by the positive x- and y-axes. Write down (but do not attempt to evaluate) a definite integral to represent the area.

This integral is evaluated in Section 3.

Solutions are given on page 50.

It is important to realise that although it is not always meaningful to calculate the area under the graph of a continuous function, it is always possible to find the definite integral of a continuous function over any interval in its domain. We evaluate any such integral $\int_a^b f$ as follows.

- Find *any* antiderivative F of the function f.
- Evaluate $[F(x)]_a^b$.

In Subsection 1.3 you will see that there is still a relationship between a definite integral and areas, even when the integrand takes negative values, but the relationship is less straightforward than in the non-negative case.

For example, to evaluate the definite integral of $f(x) = x^3$ from -2 to 1 we observe that f has antiderivative $F(x) = \frac{1}{4}x^4$ and write

$$\int_{-2}^{1} x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{-2}^{1} = \tfrac{1}{4}(1)^4 - \tfrac{1}{4}(-2)^4 = -3.75.$$

As a negative number, this cannot be an area, and yet it is a perfectly acceptable (and correct) value for the definite integral.

If an expression contains a variable that can be exchanged for another without affecting the value of the expression, then the variable is said to be a *dummy variable.*

Since a definite integral is the difference in the values of an antiderivative at the two limits of integration, the choice of variable has no bearing on the outcome. Thus each of the following definite integrals has the same value:

$$\int_0^1 e^t\,dt, \quad \int_0^1 e^x\,dx \quad \text{and} \quad \int_0^1 e^u\,du.$$

Activity 1.5 Evaluating definite integrals

Calculate to two decimal places the value of each of the following definite integrals.

(a) $\displaystyle\int_0^1 \sin(\pi x)\,dx$ (b) $\displaystyle\int_{-1}^{3} (e^{4x} + x^3)\,dx$ (c) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt$

(d) $\displaystyle\int_{-1}^{1} \frac{1}{1+u^2}\,du$ (e) $\displaystyle\int_0^1 e^t\,dt$ (f) $\displaystyle\int_{-2}^{-1} \frac{1}{x}\,dx$

Solutions are given on page 51.

There is nothing in the definition of the definite integral that requires one limit of integration to be less than the other. We can therefore examine what happens when the limits are reversed. To this end, suppose that one of the antiderivatives of f is F. Then, by definition,

The notation $\int_a^b f$ is often more convenient to use than $\int_a^b f(x)\,dx$ in general discussions.

$$\begin{aligned}\int_b^a f &= F(a) - F(b)\\ &= -(F(b) - F(a))\\ &= -\int_a^b f.\end{aligned}$$

Hence

$$\int_b^a f = -\int_a^b f,$$

so swapping the limits of integration changes the sign of the integral. Other properties concerning limits of integration can be demonstrated in a similar way.

Activity 1.6 Properties of the limits of integration

Use the definition of the definite integral to establish the following results, where a, b and c are any values in the domain of the function f.

Note that c is not an arbitrary constant here.

(a) $\displaystyle\int_a^a f = 0$

(b) $\displaystyle\int_a^b f + \int_b^c f = \int_a^c f$

Solutions are given on page 51.

For ease of reference the above rules are stated in the following box.

Properties of the limits of integration

(a) $\displaystyle\int_a^a f = 0$;

(b) $\displaystyle\int_b^a f = -\int_a^b f$;

(c) $\displaystyle\int_a^b f + \int_b^c f = \int_a^c f$.

These properties are valid whenever a, b and c lie in an interval on which the function f is continuous.

In addition, the Sum and Constant Multiple Rules apply to definite integrals just as they do to indefinite integrals, except that the multiple k no longer has to be non-zero.

Sum and Constant Multiple Rules (for definite integrals)

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{(Sum Rule)};$$

$$\int_a^b (kf) = k\int_a^b f \quad \text{(Constant Multiple Rule)}.$$

Note that k can be zero here since each side is then zero.

As before, when using either of these rules, the integrands must be continuous on a common interval. Moreover, a and b must lie in that interval.

To establish the Constant Multiple Rule, suppose that F is an antiderivative of f. Then $(kF)' = kF' = kf$. So kF is an antiderivative of kf, and we have

$$\int_a^b (kf) = [kF]_a^b = kF(b) - kF(a) = k(F(b) - F(a)) = k\int_a^b f.$$

The Sum Rule for definite integrals can be established in a similar way.

Activity 1.7 Establishing the Sum Rule

Prove the Sum Rule for definite integration.

A solution is given on page 52.

1.3 Signed area

You have seen that if f is a continuous function that is non-negative on an interval $[a, b]$, then the area under the graph of f from a to b is equal to the definite integral of f from a to b. But what happens if the graph of f has points that lie below the x-axis — is it still possible to interpret $\int_a^b f$ as an area?

To help answer this question, let us first consider the case where f is a continuous function that takes only non-positive values throughout $[a, b]$, as illustrated in Figure 1.4(a), and consider the area of the region bounded by the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

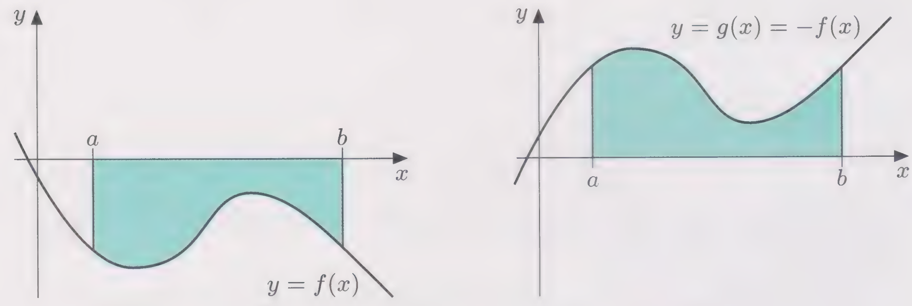

Figure 1.4 (a) Graph of f, where $f(x) \leq 0$ (b) Graph of $g(x) = -f(x)$

This case can be dealt with quickly on the basis of the previous result. If $f(x) \leq 0$ for all x in the interval $[a, b]$, then the function $g(x) = -f(x)$ satisfies the condition that $g(x) \geq 0$ for all x in the interval $[a, b]$. The graph of g is shown in Figure 1.4(b). Since this is the reflection of the graph of f in the x-axis, it follows that the area shaded beneath the graph of g is equal to the area with which we started in Figure 1.4(a). The area in Figure 1.4(a) is therefore

$$\int_a^b g(x)\,dx = \int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx.$$

So we see that, for a function f with $f(x) \leq 0$ for all x in $[a, b]$, the value of $\int_a^b f(x)\,dx$ is the negative of the area between the graph, the x-axis and the vertical lines $x = a$ and $x = b$.

Having covered the cases where $f(x)$ is either non-negative over the whole of $[a, b]$, or non-positive over all of this interval, we are left with the general situation where f may take both positive and negative values in the interval $[a, b]$. To help us understand this situation let us begin by considering the case shown in Figure 1.5, where the graph of f cuts the x-axis once at $x = c$, say.

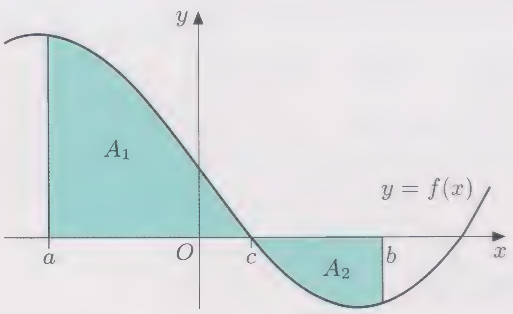

Figure 1.5 A function f that takes both positive and negative values in $[a, b]$

How can the definite integral

$$\int_a^b f$$

be related in this case to the two areas A_1 and A_2?

By limit property (c) of definite integrals we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

But $f(x) \geq 0$ for x in the interval $[a, c]$, whereas $f(x) \leq 0$ for x in the interval $[c, b]$; so, from our earlier results, it follows that

$$A_1 = \int_a^c f(x)\,dx \quad \text{and} \quad A_2 = -\int_c^b f(x)\,dx.$$

We conclude that

$$\int_a^b f(x)\,dx = A_1 - A_2. \tag{1.1}$$

It is now easy to see how this result generalises to a continuous function f whose graph cuts the x-axis any number of times. We simply have to generalise the interpretation of A_1 and A_2 as follows.

> A_1 is the sum of the areas between $x = a$ and $x = b$ that are bounded below by the x-axis, and above by the curve $y = f(x)$;
>
> A_2 is the sum of the areas between $x = a$ and $x = b$ that are bounded above by the x-axis, and below by the curve $y = f(x)$.

Once we have made these generalisations, then equation (1.1) again holds.

Example 1.1 Finding an area

Sketch, and then find, the area enclosed by the graph of $f(x) = x^2 - 1$ and the x-axis, between $x = -2$ and $x = 2$.

Solution

The area is indicated by the shading in Figure 1.6. The figure indicates that the graph crosses the x-axis at ± 1. The function f is non-negative on $[-2, -1]$ and $[1, 2]$ and non-positive on $[-1, 1]$, so the required area is

$$\begin{aligned}
A &= \int_{-2}^{-1} (x^2 - 1)\,dx - \int_{-1}^{1} (x^2 - 1)\,dx + \int_{1}^{2} (x^2 - 1)\,dx \\
&= \left[\tfrac{1}{3}x^3 - x\right]_{-2}^{-1} - \left[\tfrac{1}{3}x^3 - x\right]_{-1}^{1} + \left[\tfrac{1}{3}x^3 - x\right]_{1}^{2} \\
&= \tfrac{4}{3} - \left(-\tfrac{4}{3}\right) + \tfrac{4}{3} \\
&= 4.
\end{aligned}$$

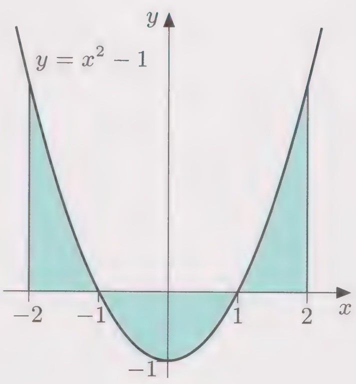

Figure 1.6 Graph of $y = x^2 - 1$

In the next activity you can compare the definite integral of a function with the area between its graph and the x-axis.

Activity 1.8 Finding an area

(a) Calculate the definite integral $\int_0^{\pi/2} \cos(3x)\,dx$.

(b) Sketch, and then find, the total area enclosed by the graph of $y = \cos(3x)$ and the x-axis, between $x = 0$ and $x = \pi/2$.

Solutions are given on page 52.

Activity 1.8 highlights the fact that the definite integral of a function over an interval cannot in general be interpreted as the area of a region. We can, however, interpret such an integral as a *signed area* in which regions above the x-axis are regarded as having positive areas (adding up to the quantity A_1 above) and those below the x-axis are regarded as having negative areas (adding up to the quantity $-A_2$). Equation (1.1) can then be regarded as saying that the definite integral of f from a to b is equal to the total signed area between the x-axis and the graph of f from a to b.

1.4 Fundamental Theorem of Calculus

This subsection will not be assessed.

We now resolve a difficulty that arose when the definite and indefinite integrals were defined earlier in the section. In both cases the definitions presupposed that every function that is continuous on an interval has an antiderivative on that interval. We can use the concept of signed area to show that such antiderivatives do exist.

The approach is similar to the one we used when the area-so-far function was introduced on page 10. There we considered a *non-negative* function f that is continuous on $[a, b]$ and we defined $A(x)$ to be the area under the graph of f from a to x. In order to generalise this definition so that it applies to *any* continuous function on $[a, b]$, we simply reinterpret $A(x)$ to be the *signed* area that lies between the graph of f and the x-axis from a to x, as shown in Figure 1.7.

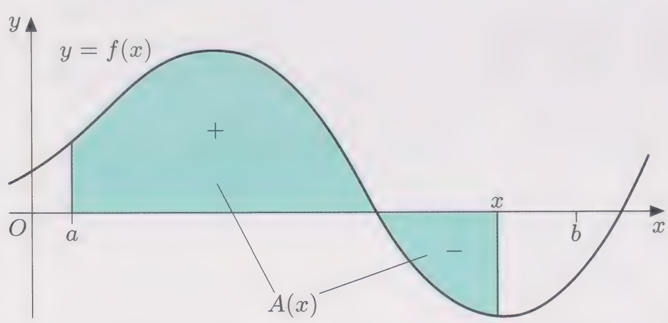

Figure 1.7 Signed-area function A

Even when A is interpreted this way, the discussion of the area-so-far function beginning on page 10 is still valid and demonstrates that

$$A'(x) = f(x).$$

So an antiderivative A of f does exist, as required.

Moreover, now that the existence of A is established, we also know that there is a whole family of antiderivatives each differing from A by a constant. Any one of these antiderivatives can then be used to define the definite and indefinite integrals, as discussed in the previous subsections.

Unfortunately the above interpretation of A falls short of a rigorous definition because it fails to explain how the various areas between the graph and the x-axis are defined. It relies on our intuitive notions about areas. This would be fine in the case of a rectangular shaped graph where we would multiply the width of the rectangle by its height, but how can we define an area bounded by a 'curved' graph?

To see how this is done consider the signed area $A(b)$ that lies between the graph of f, the x-axis and the lines $x = a$ and $x = b$, as shown in Figure 1.8. Since we know how to define the area of a rectangle we first approximate $A(b)$ by the area of a collection of N rectangles placed side by side so as to occupy approximately the same region. Each rectangle has a vertical left-hand edge that starts at the x-axis and ends at the graph. All the rectangles have width $h = (b - a)/N$.

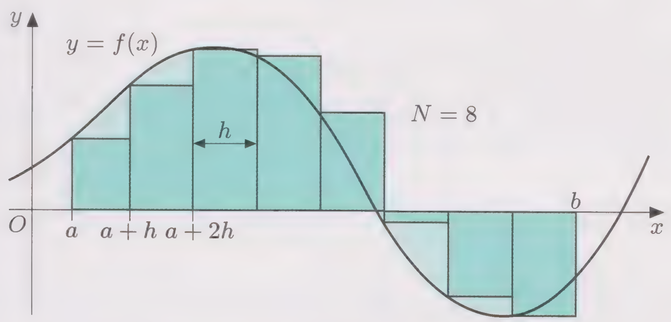

Figure 1.8 Using rectangles to approximate the signed area $A(b)$

Now if $f(a) > 0$, as shown, then the first rectangle lies above the x-axis and therefore has the (positive) signed area $hf(a)$. Whereas, if $f(a) < 0$, then the first rectangle lies below the x-axis and therefore has the (negative) signed area $hf(a)$. Either way (or if $f(a) = 0$) the signed area is given by $hf(a)$. Similarly, the second rectangle has signed area $hf(a + h)$, the third has signed area $hf(a + 2h)$, and so on. In general the $(i + 1)$th rectangle has signed area $hf(a + ih)$, and the final rectangle has signed area $hf(a + (N - 1)h)$. The total signed area therefore provides us with the following approximation to $A(b)$:

$$A(b) \simeq hf(a) + hf(a + h) + \cdots + hf(a + (N - 1)h).$$

This approximation can be written more concisely, using sigma notation, as

Sigma notation was introduced in MST121 Chapter B1, Subsection 1.2.

$$A(b) \simeq h \sum_{i=0}^{N-1} f(a + ih), \quad \text{where } h = \frac{b - a}{N}. \tag{1.2}$$

An improved approximation can be obtained by increasing the number of rectangles. We could for example double the number of rectangles by halving their width. The error which is acknowledged in writing '$\simeq$' in equation (1.2) would then be reduced. By successively doubling the number of rectangles and recalculating the sum on the right-hand side, we obtain better and better approximations to $A(b)$. Using this approach we can approximate the signed area $A(b)$ to any desired degree of accuracy.

Note that $h \to 0$ as $N \to \infty$.

Another way of putting this is to say that the *limit* as N tends to infinity of the sum on the right-hand side of equation (1.2) is equal to $A(b)$; that is,

$$A(b) = \lim_{N\to\infty}\left(h\sum_{i=0}^{N-1} f(a+ih)\right), \quad \text{where } h = \frac{b-a}{N}.$$

To be rigorous this definition should be accompanied by a proof that the limit exists for any continuous function f.

Now, we can adopt this limit as the *definition* of the signed area $A(b)$. Moreover, by allowing the right-hand edge of the area to vary, we can replace b by the variable x and thereby obtain a definition of the antiderivative $A(x)$. This in turn provides us with a whole family of antiderivatives, each differing from $A(x)$ by a constant. Any one of these antiderivatives $F(x)$ can then be used to define the definite and indefinite integrals

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{and} \quad \int f(x)\,dx = F(x) + c.$$

as described earlier in the section. Moreover, since $\int_a^b f$ is equal to $A(b) - A(a) = A(b)$, we obtain the following theorem which captures the close connection between indefinite integrals (antiderivatives) and definite integrals (limits of sums of signed areas).

Fundamental Theorem of Calculus

If f is a function which is continuous on the interval $[a, b]$, then it has an antiderivative F over $[a, b]$, and

$$\int_a^b f = F(b) - F(a) = \lim_{N\to\infty}\left(h\sum_{i=0}^{N-1} f(a+ih)\right), \tag{1.3}$$

where the variables N and h are related by the equation $h = \dfrac{b-a}{N}$.

These symbols were introduced by Leibniz in his accounts of calculus, published in 1684 and 1686.

In this context the symbol δ is used to indicate a 'small' increase in the variable that follows it.

Equation (1.3) provides us with an insight into the origins of the symbols $\int$ and dx used to represent the integration process. The expression in the large brackets on the right can be rewritten in the form

$$\sum_{i=0}^{N-1} f(x_i)\delta x,$$

where $x_i = a + ih$ are N equally spaced points a distance $\delta x = h$ apart. Historically the limiting process was indicated by replacing δx by the 'infinitesimal quantity' dx, the x_i's by the variable x, and the $\sum$ by $\int$ (an elongated s for 'limiting' sum). Overall this leads to the familiar $\int f(x)dx$ notation. This also helps to explain the word *integration* as the accumulation, or integration, of the individual 'area elements $f(x_i)\delta x$' into a whole.

The remaining text in this section is associated with Video Band C(iii), which sheds new light on the Fundamental Theorem of Calculus by using the idea of a direction field; see Figure 1.9.

Direction fields were discussed in MST121 Chapter C3.

A **direction field** for the equation $F' = f$, where f is a known function, consists of short line segments that indicate the gradients of any antiderivative F of the function f at particular points (x, y). The set of points chosen usually consists of a rectangular grid, as in Figure 1.9, with the line segments centred at those points.

Since $F'(x) = f(x)$, the gradients are given by the values of f. Thus, to ensure that $F'(x) = f(x)$, the line segment centred at (x, y) is drawn so that its gradient is equal to $f(x)$. Any antiderivative F then has gradients that are aligned with the direction field, as illustrated in Figure 1.9. The graph $y = F(x)$ in the figure is that of the unique antiderivative that passes through the point $(a, F(a))$.

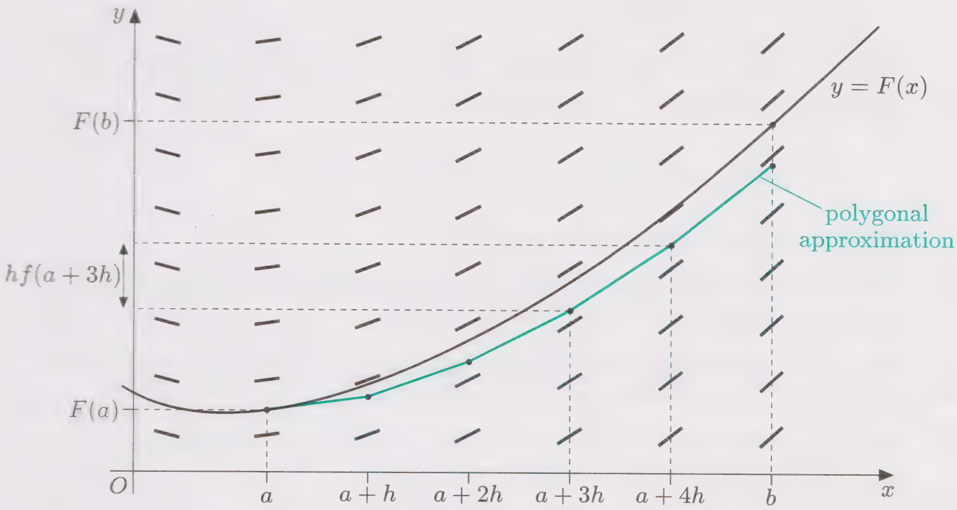

Figure 1.9 Polygonal approximation to an antiderivative F of f

A **polygonal approximation** to F is constructed as follows. First we divide the x-axis from a to b into N equal segments each of length h. At the point $(a, F(a))$ the direction field has gradient $f(a)$, so the approximation follows a line with gradient $f(a)$ until it reaches the vertical line $x = a + h$. It then changes direction, so as to realign with the direction field; that is, it follows a line with gradient $f(a + h)$. On reaching the line $x = a + 2h$ it changes direction again and follows a line with gradient $f(a + 2h)$. This process continues until the approximation reaches the line $x = b$.

For example, in Figure 1.9, $N = 5$ and so $h = (b - a)/5$.

As x increases from a to b, the y-coordinates of the graph of F change by $F(b) - F(a)$. The corresponding change in the y-coordinates of the polygonal approximation is the accumulation of the changes that occur along each straight edge of the approximation. These are each given by h multiplied by the gradient of the edge. Thus as x increases from a to $a + h$ the y-coordinate increases by $hf(a)$, as x increases from $a + h$ to $a + 2h$ the y-coordinate increases by $hf(a + h)$, and so on. Hence

$$F(b) - F(a) \simeq h \sum_{i=0}^{N-1} f(a + ih).$$

As we increase the number of edges that make up the polygonal approximation, we increase the accuracy of this approximation. In the limit as $N \to \infty$ (or as $h \to 0$) the approximation becomes an equality, as stated in the Fundamental Theorem of Calculus.

This polygonal approximation forms the basis of a numerical integration technique, known as *Euler's method*, which can be used to tabulate approximate values for antiderivatives of a function.

Now watch the band C(iii) 'Integration'.

Summary of Section 1

This section has introduced or reviewed:

◇ the concept of an antiderivative (or integral) of a function f, that is, any function whose derived function is f;

◇ the indefinite integral of a function f,

$$\int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant and F is any antiderivative of f;

◇ the definite integral of a function f,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where F is any antiderivative of f;

◇ the use of the table of standard integrals to determine definite and indefinite integrals;

◇ the use of the definite integral $\int_a^b f(x)\,dx$ to calculate the (signed) area between the graph of f and the x-axis from $x = a$ to $x = b$;

◇ the use of the Sum Rule and Constant Multiple Rule to help determine definite and indefinite integrals;

◇ the Fundamental Theorem of Calculus.

Exercises for Section 1

Exercise 1.1

Find each of the following indefinite integrals.

(a) $\int (x^{18} + x)\,dx$ (b) $\int \frac{-6}{x^4}\,dx$ (c) $\int \sin\left(\frac{1}{3}x\right)\,dx$

(d) $\int e^{-2x}\,dx$ (e) $\int (\sec^2 x + \operatorname{cosec}^2 x)\,dx$

(f) $\int \frac{(3x^2+1)(x-1)}{\sqrt{x}}\,dx$

Exercise 1.2

Calculate each of the following definite integrals.

(a) $\int_0^1 7x^8\,dx$ (b) $\int_0^{\pi/4} \cos(2x)\,dx$ (c) $\int_{-1}^1 e^{7x}\,dx$

(d) $\int_{\pi/4}^{\pi/3} \operatorname{cosec}^2 x\,dx$ (e) $\int_0^{\pi/4} \sin^2 x\,dx$

Give your answers to parts (c), (d) and (e) to two decimal places.

Exercise 1.3

(a) Find the area between the curve $y = x^2 - 4x$ and the x-axis from $x = -1$ to $x = 0$.

(b) Find the area between the curve $y = x^2 - 4x$ and the x-axis from $x = 0$ to $x = 2$.

(c) Evaluate $\int_{-1}^2 (x^2 - 4x)\,dx$.

(d) Explain why the sum of the areas found in parts (a) and (b) is not equal to $\int_{-1}^2 (x^2 - 4x)\,dx$.

2 Integration by parts

So far you have seen how the Sum and Constant Multiple Rules are used to find integrals, such as $\int(2\sin(3x) - 5\sec^2(2x) + 4x^2 + 4e^{2x})\,dx$, in which the integrand is a sum whose terms consist of multiples of functions drawn from the table of standard integrals.

Unfortunately many functions are not of this form, so in this section and the next, the class of functions that you will be able integrate is enlarged. This is achieved by the introduction of two more techniques of integration. Each technique is based on the idea of 'undoing' one of the rules of differentiation. When applying these techniques it is always useful to remember that any expression you obtain for an indefinite integral can be checked by differentiation. The technique introduced in this section is based on the idea of 'undoing' the Product Rule for differentiation.

2.1 Adapting the Product Rule

The Product Rule for differentiation states that if f and g are smooth functions, then the derivative of their product is given by

See Chapter C1, Subsection 2.2.

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x))\,g(x) + f(x)\frac{d}{dx}(g(x)).$$

For example, if $f(x) = x$ and $g(x) = \sin x$, then we can deduce from the Product Rule that

$$\begin{aligned}\frac{d}{dx}(x\sin x) &= \frac{d}{dx}(x)\sin x + x\frac{d}{dx}(\sin x)\\ &= \sin x + x\cos x.\end{aligned}$$

By rearranging this equation we can write

$$x\cos x = \frac{d}{dx}(x\sin x) - \sin x,$$

which we can then integrate, using the Sum and Constant Multiple Rules, to obtain

$$\int x\cos x\,dx = \int \frac{d}{dx}(x\sin x)\,dx - \int \sin x\,dx.$$

The first integrand on the right has antiderivative $x\sin x$ and the second has antiderivative $-\cos x$, so we have shown that

$$\int x\cos x\,dx = x\sin x + \cos x + c,$$

where c is an arbitrary constant. So knowledge of the derivative of the product $x\sin x$ has enabled us to find the indefinite integral of a different product, $x\cos x$.

To see how such an approach might work more generally, we retrace the steps of the argument above which led to finding $\int x\cos x\,dx$, but this time retain full generality, with $f(x)$ in place of x and $g(x)$ in place of $\sin x$. The starting point is the Product Rule in the form

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

or equivalently

$$(fg)' = f'g + fg'.$$

$(fg)'$ denotes the derived function of the product fg.

By rearranging this equation we obtain

$$fg' = (fg)' - f'g,$$

which we can then integrate, using the Sum and Constant Multiple Rules:

$$\int fg' = \int (fg)' - \int f'g.$$

Since $(fg)'$ has antiderivative fg, this equation can be written

$$\int fg' = fg - \int f'g.$$

Here there is no need to add an arbitrary constant to fg because one is already implicit in the indefinite integral $\int f'g$.

The formula obtained above is the basis of a technique of integration known as *integration by parts.*

In Leibniz notation, this formula is written $\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}\,v\,dx$, where $u = f(x)$ and $v = g(x)$.

Integration by parts formula (for indefinite integrals)

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{2.1}$$

This formula applies whenever all the functions involved are continuous on an interval. All the examples you will meet in this chapter are of this type.

This integration by parts formula converts the problem of integrating fg' into the problem of integrating $f'g$. If, for particular functions f and g, it is possible to find $\int f'g$, then equation (2.1) enables us to find $\int fg'$.

Example 2.1 Applying integration by parts

Use integration by parts to find the indefinite integral $\displaystyle\int x\sin x\,dx$.

Solution

To apply equation (2.1), we need to choose functions f and g' so that

$$\int x\sin x\,dx = \int f(x)g'(x)\,dx.$$

This is achieved if we take

$$f(x) = x \quad\text{and}\quad g'(x) = \sin x.$$

This choice of f and g' implies that

$$f'(x) = 1 \quad\text{and}\quad g(x) = -\cos x.$$

We could have taken

$$g(x) = -\cos x + 1$$

or indeed

$$g(x) = -\cos x + k$$

for any value of the constant k. However, we are at liberty to choose the most convenient form for $g(x)$.

Substituting into the formula for integration by parts (equation (2.1))

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

we obtain

$$\begin{aligned}\int x\sin x\,dx &= x(-\cos x) - \int 1(-\cos x)\,dx\\ &= -x\cos x + \int \cos x\,dx\\ &= -x\cos x + \sin x + c,\end{aligned}$$

where c is an arbitrary constant.

The example illustrates how important it is to choose $f(x)$ and $g'(x)$ appropriately. Had we chosen $f(x) = \sin x$ and $g'(x) = x$, then the new integrand produced by the formula in (2.1) would have been $-\frac{1}{2}x^2 \cos x$, and this would have been worse than the original integrand, $x \sin x$.

Some guidance on how to choose $f(x)$ and $g'(x)$ appropriately is given after the next activity. You can tackle both parts of this activity by first choosing $f(x) = x$, and then choosing $g'(x)$ according to this choice of f.

Activity 2.1 Using the integration by parts formula

Use integration by parts to find each of the following indefinite integrals.

(a) $\displaystyle\int x \cos x \, dx$ (b) $\displaystyle\int xe^x \, dx$

Solutions are given on page 52.

You have already seen the integral in part (a) on page 23. Here you will see how the integration by parts formula produces the result.

As observed earlier, the formula used for integration by parts converts the problem of integrating the product fg' into the problem of integrating $f'g$. To apply the technique, therefore, we must first recognise that we are integrating a product. Then this product must be expressed as fg' by choosing which part of the product to call f and which g'. In making this choice, there are two guiding principles.

◇ You must choose as g' a function for which you can find g; that is, you must be able to integrate your chosen function g'.

◇ The resulting integral $\int f'g$, obtained from using the formula, should be simpler than the integral with which you started.

There are slight variations to the second principle, which are discussed in the next subsection.

For example, in the case of Example 2.1 both factors of the integrand $x \sin x$ are easy to integrate, so it is the second guiding principle that provides the clue how to proceed. As already mentioned, had we chosen $f(x) = \sin x$ and $g'(x) = x$, then the new integrand would have been worse than the integrand we started with. The second guiding principle therefore indicates that we should choose $f(x) = x$ and $g'(x) = \sin x$. By contrast, in the next activity you should find that the first of the two guiding principles is more useful.

Activity 2.2 Choosing f and g

(a) Use integration by parts to find

$$\int x \ln x \, dx.$$

(b) Use your solution to part (a) to evaluate the definite integral

$$\int_1^2 x \ln x \, dx,$$

giving your answer to four decimal places.

Notice that $x \ln x$ is defined for all $x \in [1, 2]$.

Solutions are given on page 52.

The next activity illustrates how to perform an integration where the integrand has the form $x \cos(ax)$ or $x \sin(ax)$. In these cases it is the second guiding principle that indicates how to proceed, as in Example 2.1.

Activity 2.3 Further products with trigonometric functions

Use integration by parts to find each of the following indefinite integrals.

(a) $\displaystyle\int x\cos(2x)\,dx$ (b) $\displaystyle\int x\sin(3x)\,dx$

Solutions are given on page 52.

Once you have used integration by parts to find an indefinite integral, then you can find a corresponding definite integral by applying the limits of integration in the usual way (as in Activity 2.2(b)). The next activity uses this approach to determine the area under a graph.

Activity 2.4 Integration by parts to find an area

(a) Use integration by parts to find

$$\int xe^{2x}\,dx.$$

(b) Explain why the graph of $y = xe^{2x}$ is above the x-axis for all values of x in the interval $[1, 2]$, and write down a definite integral which gives the area under this graph from $x = 1$ to $x = 2$.

(c) Use your result from part (a) to evaluate the area described in part (b), giving your answer to four decimal places.

Solutions are given on page 53.

When finding a definite integral using integration by parts, it is sometimes more convenient to apply the limits 'as you go along'. You can do this using the following result.

Integration by parts formula (for definite integrals)

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx. \qquad (2.2)$$

For comparison, this result is used in the following example to recalculate the definite integral that you found in Activity 2.4(c).

Example 2.2 Checking a previous result

Use equation (2.2) to evaluate the definite integral

$$\int_1^2 xe^{2x}\,dx.$$

Give your answer to four decimal places.

Solution

Let $f(x) = x$ and $g'(x) = e^{2x}$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = \tfrac{1}{2}e^{2x}.$$

Substituting into equation (2.2), we obtain

$$\begin{aligned}\int_1^2 xe^{2x}\,dx &= \left[x\tfrac{1}{2}e^{2x}\right]_1^2 - \int_1^2 \tfrac{1}{2}e^{2x}\,dx\\ &= \left(e^4 - \tfrac{1}{2}e^2\right) - \tfrac{1}{2}\left[\tfrac{1}{2}e^{2x}\right]_1^2\\ &= \left(e^4 - \tfrac{1}{2}e^2\right) - \left(\tfrac{1}{4}e^4 - \tfrac{1}{4}e^2\right)\\ &= \tfrac{3}{4}e^4 - \tfrac{1}{4}e^2\\ &= 39.1013 \text{ (to 4 d.p.)},\end{aligned}$$

as before.

A similar approach can be used to evaluate the definite integral in the following activity.

Activity 2.5 Integration by parts for a definite integral

Use the integration by parts formula for definite integrals to find

$$\int_0^1 xe^{3x}\,dx,$$

giving your answer to four decimal places.

A solution is given on page 53.

The next activity is like Activity 2.4 but it asks you to calculate the area directly using equation (2.2) rather than first calculate the indefinite integral.

Activity 2.6 Finding an area using integration by parts

(a) Explain why the graph of $y = x\sin(3x)$ is above the x-axis for values of x in the interval $[\pi/12, \pi/6]$, and write down a definite integral which gives the area under this graph from $x = \pi/12$ to $x = \pi/6$.

(b) Use the integration by parts formula for definite integrals to evaluate the area described in part (a), giving your answer to four decimal places.

Solutions are given on page 53.

2.2 Extending the method

This subsection demonstrates how the method of integration by parts, for finding both indefinite and definite integrals, can be extended to products of functions other than those considered so far.

All our earlier products had x as one of the factors, and in all but one case, it was appropriate to choose $f(x) = x$ in the integration by parts formula. The exception was in Activity 2.2, where for the product $x \ln x$ it is the choice $f(x) = \ln x$ which leads to a simpler integral. A similar approach sometimes works also for other products of $\ln x$ and a power of x.

Activity 2.7 Products involving a logarithm

Find each of the following integrals.

(a) $\displaystyle\int x^3 \ln x \, dx$

(b) $\displaystyle\int \ln x \, dx$ (*Hint*: Note that $\ln x$ can be written as $\ln x \times 1$.)

Solutions are given on page 53.

Integration by parts is a technique which allows us to integrate certain products of functions. Activity 2.7(b) is a reminder that *any* function may be regarded as a product, namely, the product of the original function and the constant function with value 1. This idea, applying the integration by parts formula with $g'(x) = 1$, can be useful on other occasions also.

Activity 2.8 Integrating arctan

See Subsection 3.1, Frame 5.

In the next section you will see that $\int x/(1+x^2)\, dx = \frac{1}{2}\ln(1+x^2) + c$, where c is an arbitrary constant. Use this result to find the indefinite integral

$$\int \arctan x \, dx.$$

A solution is given on page 54.

Another kind of product that we can sometimes integrate using the integration by parts formula has x^2 (rather than x, $\ln x$ or 1) as one of its factors.

Example 2.3 Two-step integration by parts

Find the indefinite integral $\displaystyle\int x^2 e^{4x} \, dx$.

Solution

We apply the integration by parts formula (2.1).

Let $f(x) = x^2$ and $g'(x) = e^{4x}$; then

$$f'(x) = 2x \quad \text{and} \quad g(x) = \tfrac{1}{4}e^{4x}.$$

Substituting into equation (2.1), we obtain

$$\begin{aligned}\int x^2e^{4x}\,dx &= x^2\left(\tfrac{1}{4}e^{4x}\right) - \int 2x\left(\tfrac{1}{4}e^{4x}\right)\,dx\\ &= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{2}\int xe^{4x}\,dx. \end{aligned}\qquad (2.3)$$

This equation expresses the integral we were asked to find, $\int x^2e^{4x}\,dx$, in terms of the integral $\int xe^{4x}\,dx$. Here the power of x is one less than that in the original integral.

The integral $\int xe^{4x}dx$ can itself be found using integration by parts.

Let $f(x) = x$ and $g'(x) = e^{4x}$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = \tfrac{1}{4}e^{4x}.$$

Here the function f is different from that labelled as f above, since this is a separate application of the integration by parts formula (2.1). The function g happens to be the same as before.

Substituting into equation (2.1), we obtain

$$\begin{aligned}\int xe^{4x}\,dx &= x\tfrac{1}{4}e^{4x} - \int \tfrac{1}{4}e^{4x}\,dx\\ &= \tfrac{1}{4}xe^{4x} - \tfrac{1}{16}e^{4x} + c\\ &= \tfrac{1}{16}e^{4x}(4x-1) + c,\end{aligned}$$

where c is an arbitrary constant.

By using this result to substitute for $\int xe^{4x}\,dx$ on the right-hand side of equation (2.3), we obtain

$$\begin{aligned}\int x^2e^{4x}\,dx &= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{2}\left(\tfrac{1}{16}e^{4x}(4x-1)\right) + c\\ &= \tfrac{1}{32}e^{4x}(8x^2-4x+1) + c,\end{aligned}$$

where c is an arbitrary constant.

You may think we should write $-\frac{1}{2}c$ here rather than $+c$ but, as usual, we absorb into c any constants that multiply, or are added to, it. The status of c as an arbitrary constant will not be affected by this.

In Example 2.3, we were able to find $\int x^2e^{4x}dx$ by using integration by parts *twice*. This enabled us to 'eliminate the x^2 term'. After a complicated calculation like this, it is advisable to verify your answer by differentiating it. You might like to check that

$$\frac{d}{dx}\left(\tfrac{1}{32}e^{4x}(8x^2-4x+1) + c\right)$$

is equal to x^2e^{4x}.

Activity 2.9 Using integration by parts twice

Find each of the following integrals.

(a) $\displaystyle\int x^2e^x\,dx$ (*Hint*: You found $\int xe^x dx$ in Activity 2.1(b).)

(b) $\displaystyle\int x^2\cos(3x)\,dx$ (*Hint*: You found $\int x\sin(3x)\,dx$ in Activity 2.3(b).)

Solutions are given on page 54.

As a final extension to the kinds of products that can be integrated using the integration by parts formula we work through an example where a second integration by parts yields an equation that expresses the required integral in terms of itself; the integral is then found by solving this equation.

Example 2.4 Expressing an integral in terms of itself

Find the indefinite integral $\int e^x \sin(2x)\,dx$.

Solution

We apply the integration by parts formula (2.1).

We could also have chosen $f(x) = e^x$ and $g'(x) = \sin(2x)$; however $g'(x) = e^x$ is slightly easier to integrate and it avoids the introduction of fractions.

Let $f(x) = \sin(2x)$ and $g'(x) = e^x$; then

$$f'(x) = 2\cos(2x) \quad \text{and} \quad g(x) = e^x.$$

Substituting into the formula, we obtain

$$\begin{aligned}\int e^x \sin(2x)\,dx &= e^x \sin(2x) - \int e^x (2\cos(2x))\,dx \\ &= e^x \sin(2x) - 2\int e^x \cos(2x)\,dx. \qquad (2.4)\end{aligned}$$

Now the integral on the right-hand side of this equation is no simpler in form than the one with which we started. However, let us see what happens if we apply integration by parts once again, to the integral

$$\int e^x \cos(2x)\,dx.$$

As usual, we start by choosing the functions f and g.

Let $f(x) = \cos(2x)$ and $g'(x) = e^x$; then

$$f'(x) = -2\sin(2x) \quad \text{and} \quad g(x) = e^x.$$

It follows from formula (2.1) that

$$\begin{aligned}\int e^x \cos(2x)\,dx &= e^x \cos(2x) - \int e^x (-2\sin(2x))\,dx \\ &= e^x \cos(2x) + 2\int e^x \sin(2x)\,dx, \qquad (2.5)\end{aligned}$$

and this last expression involves the integral $\int e^x \sin(2x)\,dx$ with which we started!

Substituting the right-hand side of equation (2.5) into equation (2.4), we obtain

$$\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2\left(e^x \cos(2x) + 2\int e^x \sin(2x)\,dx\right);$$

that is,

$$\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x)\,dx.$$

On collecting the $\int e^x \sin(2x)\,dx$ terms on the left-hand side, we obtain

$$5\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x \cos(2x),$$

For convenience, the arbitrary constant has been omitted from this integral, but one must be included, after dividing by 5, in the next (and final) step.

which leads to the conclusion that

$$\int e^x \sin(2x)\,dx = \tfrac{1}{5}e^x \sin(2x) - \tfrac{2}{5}e^x \cos(2x) + c,$$

where c is an arbitrary constant.

(It is advisable to check the result of a complicated manipulation like this by differentiation.)

This approach of using integration by parts twice and then solving the resulting equation will work for the integration of any product of an exponential function with either a sine or a cosine function. For such examples, it does not matter how you choose the f and the g', provided that *you make the same choice on both occasions*; that is, either f is put equal to the exponential function both times or f is put equal to the trigonometric function both times. If you do not do this, then you will obtain an equation that reduces to $0 = 0$.

Activity 2.10 Product of exponential and sin or cos

Find each of the following integrals.

(a) $\int e^x \cos(3x)\,dx$ (b) $\int e^{2x} \sin x\,dx$

Solutions are given on page 54.

Summary of Section 2

This section has introduced:

- the technique of integration by parts for evaluating definite and indefinite integrals;
- examples of integrating by parts where one factor of the integrand is x;
- examples of integration by parts where one factor of the integrand is $\ln x$;
- examples where integration by parts can be applied by treating the integrand as a product of itself and 1;
- examples where a second integration by parts is needed to complete the integration;
- examples where a second integration by parts yields an equation that expresses the required integral in terms of itself.

Exercise for Section 2

Exercise 2.1

Use integration by parts to find each of the following integrals.

(a) $\int xe^{-x}\,dx$ (b) $\int \sqrt{x}\ln x\,dx$ (c) $\int x\cos(4x)\,dx$

(d) $\int x^2 \sin(3x)\,dx$ (e) $\int e^{3x}\cos x\,dx$

(f) $\int_0^1 xe^{-x}\,dx$. (Use your answer to part (a), and give your answer to four decimal places.)

3 Integration by substitution

In Section 2 the rule for integration by parts was derived from the Product Rule for derivatives. In this section a rule for integration based on the Composite Rule (or Chain Rule) for differentiating composites of functions is derived. This rule extends the types of integrals that you will be able to solve.

3.1 Adapting the Composite Rule

Recall that the derivative of the composite $f(g(x))$ is obtained by differentiating the outer function f, leaving the inner function unchanged, and then multiplying this by the derivative of the inner function g. That is

See Chapter C1, Subsection 2.3, where the Composite Rule was stated for the composite $g(f(x))$.

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x). \tag{3.1}$$

If we can recognise that a given function has the form shown on the right of equation (3.1), then we know that it has antiderivative $f(g(x))$. In the first of the tape frames that follow we begin with an example that illustrates how this observation can help us to integrate certain functions.

Now listen to CDA5494 (Tracks 7–10), band 2, 'Integration by substitution'.

Frame 1

Integration by substitution

By the Chain Rule,

$$\frac{d}{dx}\left(\sin(x^3)\right) = \cos(x^3) \times 3x^2$$
$$= 3x^2 \cos(x^3)$$

so

$$\int 3x^2 \cos(x^3)\,dx = \sin(x^3) + c. \quad (1)$$

Let $u = x^3$; then

LHS of (1) $\quad \int 3x^2 \cos(x^3)\,dx = \int \overbrace{\cos(x^3)}^{\cos u}\overbrace{(3x^2)}^{\frac{du}{dx}}\,dx = \int \cos u \frac{du}{dx}\,dx$

RHS of (1) $\quad \sin(x^3) + c = \sin u + c = \int \cos u\,du$

so

$$\int \cos u \frac{du}{dx}\,dx = \int \cos u\,du.$$

If $f(u) = \cos u$, then (1) takes the form:

$$\int f(u) \frac{du}{dx}\,dx = \int f(u)\,du. \quad (2)$$

Generalising:

Integration by substitution

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \text{ where } u = g(x). \quad (3)$$

f continuous
g smooth

replace $g(x)$ by u
replace $\frac{du}{dx}\,dx$ by du

Proof

Let $F(u)$ be an antiderivative of $f(u)$, that is $F'(u) = f(u)$. Then

$$\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x)$$
$$= f(g(x))g'(x),$$

so

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + c = F(u) + c = \int f(u)\,du.$$

Frame 2

Strategy to find $\int f(g(x))g'(x)dx$ or a constant multiple

Step 1 Choose $u = g(x)$ and find $\frac{du}{dx} = g'(x)$.

Step 2 Substitute $u = g(x)$ into $f(g(x))$ and replace $g'(x)dx$ by du.

Step 3 Find $\int f(u)\,du$.

Step 4 Substitute back for u in terms of x.

Frame 3

Example

Find $\int x^2(x^3+1)^8\,dx$, taking $u = x^3 + 1$.

Step 1 Choose $u = x^3 + 1$; then $\frac{du}{dx} = 3x^2$.

Step 2
$$\begin{aligned}\int x^2\left(x^3+1\right)^8 dx &= \tfrac{1}{3}\int\left(x^3+1\right)^8\left(3x^2\right)dx\\ &= \tfrac{1}{3}\int u^8\frac{du}{dx}\,dx\\ &= \tfrac{1}{3}\int u^8\,du\end{aligned}$$

Rearrange, adjust by 3 and compensate.

du replaces $3x^2\,dx$.

Step 3 $\frac{1}{3}\int u^8\,du = \frac{1}{3} \times \frac{1}{9}u^9 + c = \frac{1}{27}u^9 + c$

Step 4 $\int x^2(x^3+1)^8\,dx = \frac{1}{27}(x^3+1)^9 + c$

Frame 4

Activity 3.1

Use integration by substitution to find each of the following integrals.

(a) $\int (5+2x^2)^{16}x\,dx$, taking $u = 5 + 2x^2$

(b) $\int x^2\sec^2(x^3)\,dx$, taking $u = x^3$

(c) $\int (\sin x)e^{\cos x}\,dx$, taking $u = \cos x$

(d) $\int \frac{1}{1+9x^2}\,dx$, taking $u = 3x$

Solutions are given on page 55.

Frame 5

Example

Find $\int \frac{x}{1+x^2}\,dx$, taking $u = 1 + x^2$.

With $u = 1 + x^2$, we have $\frac{du}{dx} = 2x$. Thus

Step 1

$$\begin{aligned}\int \frac{x}{1+x^2}\,dx &= \tfrac{1}{2}\int \frac{1}{1+x^2}(2x)\,dx \\ &= \tfrac{1}{2}\int \frac{1}{u}\,du \\ &= \tfrac{1}{2}\ln|u| + c \\ &= \tfrac{1}{2}\ln(1+x^2) + c.\end{aligned}$$

Step 2: rearrange, adjust by 2 and compensate, and replace $2x\,dx$ by du.

Step 3

Step 4

$1 + x^2 > 0$

Frame 6

Activity 3.2

Remember: $-\ln(a) = \ln(1/a)$.

Find $\int \tan x\,dx$, taking $u = \cos x$.

A solution is given on page 55.

Frame 7

General result

On intervals where $f(x) \neq 0$: $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$

Substitute $u = f(x)$, replace $f'(x)\,dx$ by du; then $\int \frac{1}{u}\,du = \ln|u| + c$.

Frame 8

Activity 3.3

By using an appropriate substitution, find each of the following integrals.

(a) $\int (6+x^3)^9 x^2\,dx$

(b) $\int e^{2x}\sin(e^{2x})\,dx$

(c) $\int \sin^4 x \cos x\,dx$

(d) $\int x\sqrt{1+x^2}\,dx$

(e) $\displaystyle\int \frac{x^2}{1+x^3}\,dx$

Solutions are given on page 55.

Frame 9

Integration by substitution – definite integrals

If f has antiderivative F, then

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du,$$

so

$$\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \text{ where } u = g(x).}$$

$\frac{d}{dx}F(g(x))$
$= F'(g(x))g'(x)$
$= f(g(x)g'(x)$

Frame 10

Example

Find $\int_0^1 e^{2x}\sqrt{1+e^{2x}}\,dx$.

Choose $u = 1 + e^{2x}$; then $\frac{du}{dx} = 2e^{2x}$. So

$$\begin{aligned}\int_0^1 e^{2x}\sqrt{1+e^{2x}}\,dx &= \tfrac{1}{2}\int_0^1 \sqrt{1+e^{2x}}\,(2e^{2x})\,dx\\ &= \tfrac{1}{2}\int_2^{1+e^2}\sqrt{u}\,du\\ &= \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_2^{1+e^2}\\ &= \tfrac{1}{3}\left((1+e^2)^{3/2} - 2^{3/2}\right) = 7.1565 \text{ (to 4 d.p.)}.\end{aligned}$$

If $x = 0$, then $u = 1 + e^{2\times 0} = 2$;
if $x = 1$, then $u = 1 + e^{2\times 1} = 1 + e^2$.

Frame 11

Activity 3.4

Use integration by substitution to evaluate each of the following integrals to four decimal places.

(a) $\int_0^1 \frac{x}{1+2x^2}\,dx$, taking $u = 1 + 2x^2$

(b) $\int_0^{\pi/2} \frac{\sin x \cos x}{1+\cos^2 x}\,dx$, taking $u = 1 + \cos^2 x$

(c) $\int_{-1}^0 xe^{-3x^2-2}\,dx$, taking $u = -3x^2 - 2$

(d) $\int_0^{\pi/4} \sec^3 x \tan x\,dx$, taking $u = \sec x$

Solutions are given on page 56.

3.2 Extending the method

So far we have used the method of integration by substitution in cases where the integrand can be expressed as a product of two terms in which one term is the derivative of an expression present in the other. In some cases the product involves an additional expression in x that will need to be expressed in terms of u. The next example illustrates this point.

Example 3.1 Expressing x in terms of u

Find the indefinite integral

$$\int x^2\sqrt{\tfrac{1}{2}x+3}\,dx.$$

Solution

To simplify the square-root term we put $u = \frac{1}{2}x + 3$; then

$$\frac{du}{dx} = \tfrac{1}{2} \quad \text{and} \quad x = 2u - 6.$$

Thus

$$\begin{aligned}\int x^2\sqrt{\tfrac{1}{2}x+3}\,dx &= \int 2x^2\sqrt{\tfrac{1}{2}x+3}\,(\tfrac{1}{2})\,dx \\ &= \int 2(2u-6)^2\sqrt{u}\,du \\ &= \int (8u^{5/2} - 48u^{3/2} + 72u^{1/2})\,du \\ &= \tfrac{16}{7}u^{7/2} - \tfrac{96}{5}u^{5/2} + 48u^{3/2} + c,\end{aligned}$$

Here the 'additional expression in x' is x^2.

where c is an arbitrary constant.

Finally, substituting back for u in terms of x gives the complicated answer

$$\int x^2\sqrt{\tfrac{1}{2}x+3}\,dx = \tfrac{16}{7}(\tfrac{1}{2}x+3)^{7/2} - \tfrac{96}{5}(\tfrac{1}{2}x+3)^{5/2} + 48(\tfrac{1}{2}x+3)^{3/2} + c.$$

The integrals in the next activity can be found by expressing x in terms of u in a similar way. In each case the purpose of the given substitution is to 'get rid of' an awkward term, such as an inconvenient square root or denominator.

Activity 3.5 Putting x in terms of u

Use integration by substitution to find each of the following indefinite integrals.

(a) $\displaystyle\int \frac{x}{(1+2x)^3}\,dx$, taking $u = 1 + 2x$

(b) $\displaystyle\int x\sqrt{1+x}\,dx$, taking $u = 1 + x$

(c) $\displaystyle\int \frac{2x+3}{(x+2)^{1/3}}\,dx$, taking $u = x + 2$

Solutions are given on page 56.

Up to now we have tried to simplify a given integral by replacing $(du/dx)\,dx$ by du, but this depends on there being a suitable candidate for du/dx in the integrand. Consider the integral $\int(3+\sqrt{x})^9\,dx$, and the substitution $u = 3+\sqrt{x}$. Then

$$du/dx = \tfrac{1}{2}x^{-1/2},$$

but there is no suitable candidate for du/dx in the integrand. An alternative approach in such circumstances is to perform a so-called *'backwards substitution'* in which dx is replaced by $(dx/du)\,du$, as in the following example.

The justification for this approach is given at the end of the subsection.

Example 3.2 Backwards substitution

Use backwards substitution to find the indefinite integral

$$\int (3+\sqrt{x})^9\,dx.$$

Solution

To simplify $(3+\sqrt{x})^9$ we put $u = 3+\sqrt{x}$; then

$$x = (u-3)^2, \quad \text{so} \quad \frac{dx}{du} = 2(u-3).$$

By substituting for x in terms of u, and replacing dx by $\dfrac{dx}{du}\,du = 2(u-3)\,du$, we obtain

$$\begin{aligned}\int (3+\sqrt{x})^9\,dx &= \int u^9 \times (2(u-3))\,du \\ &= \int (2u^{10} - 6u^9)\,du \\ &= \frac{2u^{11}}{11} - \frac{6u^{10}}{10} + c \\ &= \tfrac{2}{11}(3+\sqrt{x})^{11} - \tfrac{3}{5}(3+\sqrt{x})^{10} + c,\end{aligned}$$

where c is an arbitrary constant.

In the above example the backwards substitution $x = (u-3)^2$, where $u = 3+\sqrt{x}$, is used to make $(3+\sqrt{x})^9$ more manageable. The next activity invites you to make the denominator of the integrand more manageable by using a backwards trigonometric substitution.

Activity 3.6 Using a trigonometric substitution

Use backwards substitution to find the indefinite integral

$$\int \frac{1}{(1+x^2)^2}\,dx, \quad \text{taking } x = \tan u, \text{ where } u = \arctan x.$$

Solutions are given on page 57.

There are several points to note arising from Example 3.2 and Activity 3.6.

- ◇ In Example 3.2, it is easy to see what substitution to try: $u = 3 + \sqrt{x}$. This substitution does not enable us to find the integral but it does provide the backwards substitution $x = (u-3)^2$, its inverse, which enables us to integrate the expression.
- ◇ In Activity 3.6, the original variable x is expressed as a one-one function of the new variable u, in the form $x = h(u)$, where $u = h^{-1}(x)$. This ensures that, when using the substitution, we can move from an x-value to a u-value, and back to the same x-value.

Backwards substitution can also be used to evaluate *definite* integrals, in the usual way, by adapting the limits of integration so that they refer to the new variable u. This is illustrated in the final example of backwards substitution where we use integration to derive the result that a circle of unit radius encloses an area of π square units.

Example 3.3 Finding the area of a circle

Evaluate the definite integral

$$\int_0^1 \sqrt{1-x^2}\,dx, \quad \text{taking } x = \sin u, \text{ where } u = \arcsin x.$$

You showed in Activity 1.4(d) that this integral represents the area of a quarter-circle of radius one unit.

Solution

Putting $x = \sin u$, we obtain $dx/du = \cos u$. Also:

$$\text{if } x = 0, \text{ then } u = \arcsin(0) = 0;$$
$$\text{if } x = 1, \text{ then } u = \arcsin(1) = \tfrac{1}{2}\pi.$$

Hence

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\pi/2} \sqrt{1-\sin^2 u}\,(\cos u)\,du \\ &= \int_0^{\pi/2} \cos^2 u\,du \\ &= \int_0^{\pi/2} \tfrac{1}{2}(1+\cos(2u))\,du \\ &= \left[\tfrac{1}{2}u + \tfrac{1}{4}\sin(2u)\right]_0^{\pi/2} \\ &= \tfrac{1}{4}\pi.\end{aligned}$$

We have

$$\sqrt{1-\sin^2 u} = \sqrt{\cos^2 u} = \cos u,$$

since

$$\cos^2 u + \sin^2 u = 1$$

and

$$\cos u \geqslant 0, \text{ for } u \in [0, \pi/2].$$

This result shows that the area of a quarter-circle of radius one is $\frac{1}{4}\pi$. The area of the full circle is therefore π, as expected.

A similar argument, starting from the definite integral

$$\int_0^r \sqrt{r^2 - x^2}\,dx,$$

shows that the area of a circle of radius r is πr^2.

In the next activity the substitution is chosen so that the denominator simplifies to u. Remember to adapt the limits of integration so that they refer to this new variable u. This way you will be saved the bother of having to substitute back for u in terms of x in the indefinite integral.

Activity 3.7 Evaluating a definite integral

Calculate, to four decimal places, the definite integral

$$\int_1^4 \frac{x}{1+\sqrt{x}}\,dx, \quad \text{taking } x = (u-1)^2, \text{ where } u = 1 + \sqrt{x}.$$

Solutions are given on page 57.

In backwards substitution we replace dx by $(dx/du)\,du$, rather than replace $(du/dx)\,dx$ by du. But can this new feature be justified? In fact the justification is straightforward. Starting with formula (3) in Frame 1, we interchange the two sides of the formula, to obtain

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx, \qquad \text{where } u = g(x),$$

rename the function g to be h, and then swap the names given to the variables u and x, to obtain

$$\int f(x)\,dx = \int f(h(u))h'(u)\,du, \quad \text{where } x = h(u). \tag{3.2}$$

So when we make the substitution $x = h(u)$, it is legitimate to replace dx by $(dx/du)\,du$, because $h'(u) = dx/du$.

3.3 Choosing a method

You have now met a number of approaches to evaluating integrals. However, when given an integral, it may not be apparent which method, if any, to try. Check first whether the integral is a standard function, or a sum of multiples of standard functions, or can be rearranged as such with some algebra. If it is not, then try integration by parts, or by substitution.

Typically, for integration by parts to be effective, the integrand needs to be a product, and one term of the product should give a constant when differentiated one or more times. However, examples such as those in Activity 2.10 and Example 2.4 show that there are other situations where integration by parts can be effective.

It is most readily evident that integration by substitution is appropriate if you can recognise the integrand as being of the form

$$f(g(x)) \times g'(x),$$

where f is a function that you can integrate. In such a case, use the substitution $u = g(x)$. There are other situations where a backwards substitution of the form $x = h(u)$, where $u = h^{-1}(x)$, can help, and you have seen several examples. In this course, you are *not* expected to provide such backwards substitutions, but you should be able to carry out a given such substitution.

In the next activity you are asked to recognise which method of integration to use.

Activity 3.8 Which method of integration?

There is no need to do the integrals now. You will be invited to try evaluating them as Exercise 3.5.

You may like to think of a suitable domain for the indefinite integral in part (e).

For each integral below, suggest which method of integration to use and a suitable substitution where you choose the substitution method.

(a) $\displaystyle\int x^3\cos(5x^4)\,dx$ (b) $\displaystyle\int x\cos(5x)\,dx$ (c) $\displaystyle\int (x+\cos(5x))\,dx$

(d) $\displaystyle\int \frac{1}{1+2x^2}\,dx$ (e) $\displaystyle\int \frac{x^2}{(7-x^3)^7}\,dx$ (f) $\displaystyle\int xe^{-x/3}\,dx$

(g) $\displaystyle\int (x-1)^4\,dx$

Solutions are given on page 57.

Summary of Section 3

This section has introduced:

◇ the technique of integration by substitution for evaluating definite and indefinite integrals;

◇ the technique of backwards substitution.

Exercises for Section 3

Exercise 3.1

Use the given substitution to find each of the following integrals.

(a) $\displaystyle\int x(1+x^2)^3\,dx$, taking $u = 1+x^2$

(b) $\displaystyle\int_0^{\pi/4} \sin x\cos^3 x\,dx$, taking $u = \cos x$

(c) $\displaystyle\int \frac{x-2}{x^2-4x+3}\,dx$, taking $u = x^2-4x+3$

(d) $\displaystyle\int \frac{x^5}{1+x^3}\,dx$, taking $u = 1+x^3$

(e) $\displaystyle\int_4^9 \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx$, taking $u = \sqrt{x}-1$

Exercise 3.2

Use integration by substitution to find each of the following integrals.

(a) $\displaystyle\int_0^1 x(1+x^2)^5\,dx$ (b) $\displaystyle\int \cos x\sin^3 x\,dx$

(c) $\displaystyle\int (24x+9)(4x^2+3x+5)^7\,dx$ (d) $\displaystyle\int_0^2 \frac{6x+4}{\sqrt{3x^2+4x+16}}\,dx$

Exercise 3.3

Use the suggested (backwards) substitution to find each of the following integrals.

(a) $\displaystyle\int \frac{x}{\sqrt{x-2}}\,dx$, taking $x = u^2+2$, where $u = \sqrt{x-2}$

(b) $\displaystyle\int_0^3 \frac{1}{9+x^2}\,dx$, taking $x = 3\tan u$, where $u = \arctan(x/3)$

(c) $\displaystyle\int (1-x^2)^{-3/2}\,dx$, taking $x = \sin u$, where $u = \arcsin x$

Exercise 3.4

Use integration by substitution to find each of the following integrals.

(a) $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$ (b) $\displaystyle\int \frac{x^3}{(1+x^2)^{1/2}}\,dx$ (c) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

The substitutions are less easy to see here. If you cannot find an appropriate substitution after a reasonable time, then turn directly to the solution.

Exercise 3.5

Find each of the integrals in Activity 3.8, using any appropriate method.

4 Volumes of solids of revolution

In Subsection 1.2 we discussed how integration can be applied to the problem of finding an area beneath a curve. In this section, you will see that definite integrals can also be used to obtain the volumes of solid objects, or *solids*. The activities which arise here will also give you further practice in integration.

We concentrate on one particular class of solids, known as *solids of revolution*. A solid of this type may be constructed by rotating the region bounded by the graph of a continuous function f, the x-axis and the lines $x = a$ and $x = b$ through $360°$ about the x-axis.

Recall that a region includes its boundary.

In this way, the region sweeps out a solid. The solid is said to be 'generated by' the region (see Figure 4.1). Each cross-section of the solid in a plane perpendicular to the x-axis is circular in shape.

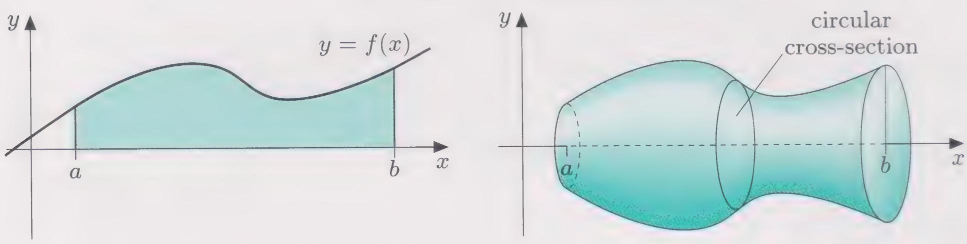

Figure 4.1 Generating a solid of revolution

For example, the outcome of rotating the triangular region shown shaded in Figure 4.2 is a cone, while rotation of the semicircular region in Figure 4.3 produces a sphere.

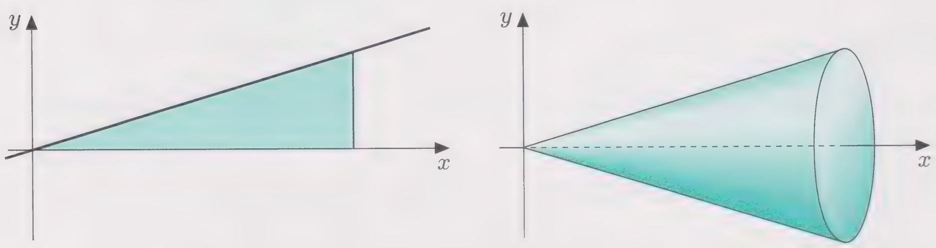

Figure 4.2 Effect of rotating a triangular region

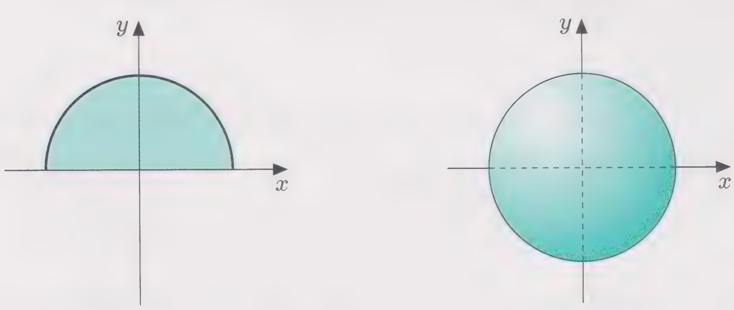

Figure 4.3 Effect of rotating a semicircular region

The simplest solid of revolution is that obtained by rotating a rectangular region, which arises when the function f is a constant function. The result is a cylinder; see Figure 4.4. If the function is $f(x) = r$, and the region is bounded as shown in Figure 4.4, then the resulting cylinder has volume

$$\pi r^2(b-a). \tag{4.1}$$

This volume is just the product of the area πr^2 of the circular cross-section and the length $b-a$ in the direction perpendicular to this cross-section.

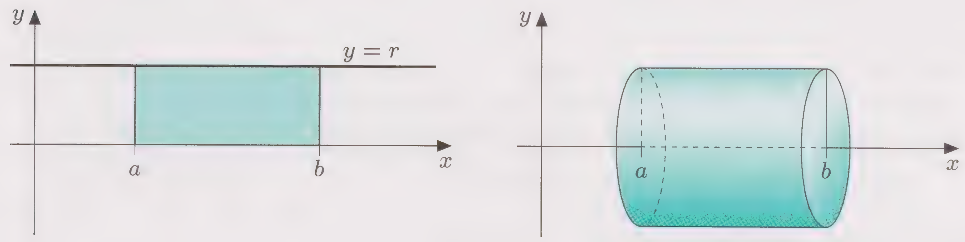

Figure 4.4 Effect of rotating a rectangle

Consider now the more general continuous function f whose graph is shown in Figure 4.1. We seek a formula for the volume of the solid of revolution obtained by rotating the region bounded by this graph, the x-axis and the lines $x = a$ and $x = b$, about the x-axis.

Let $V(x)$ be the volume of that part of this solid of revolution between a and x, where $a \le x \le b$. That is, $V(x)$ is the volume of the solid bounded by the graph of f and the planes through a and x which are perpendicular to the x-axis, as illustrated in Figure 4.5.

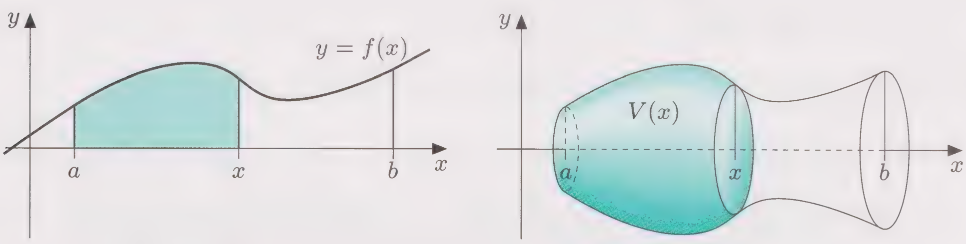

Figure 4.5 Defining $V(x)$

First, note that $V(a) = 0$, since if $x = a$ then the solid with volume $V(x)$ has zero width.

The argument here is very similar in structure to that used in the case of areas in Subsection 1.2.

Now consider also the solid with volume $V(x+h)$, for which the right-hand bounding cross-sectional plane is moved a distance h (> 0) to the right. Figure 4.6 shows the solid between the cross-sectional planes at x and at $x+h$; the volume of this solid is

$$V(x+h) - V(x).$$

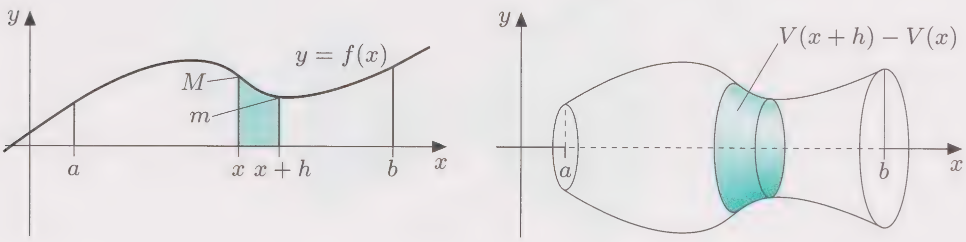

Figure 4.6 Volume $V(x+h) - V(x)$

Over the interval $[x, x+h]$, the values of the function f lie between a minimum value m and a maximum value M; see Figure 4.6. The region bounded by the graph of f and the x-axis on this interval therefore contains a rectangle of height m and is contained in a rectangle of height M.

The case for which $f(x) > 0$ on the interval $[x, x+h]$ is covered here, but, as you will see, the approach can be adapted to apply to $f(x) \leq 0$ also.

Suppose that each of these two rectangles is rotated about the x-axis to form a cylinder. The smaller cylinder, of radius m, will be contained in that part of the original solid of revolution between x and $x+h$, which in turn will be contained in the larger cylinder, of radius M. According to expression (4.1), these cylinders have volume $\pi m^2 h$ and $\pi M^2 h$, respectively, so we can write the relation between the three volumes as

$$\pi m^2 h \leq V(x+h) - V(x) \leq \pi M^2 h.$$

On dividing the inequalities by h, we obtain

$$\pi m^2 \leq \frac{V(x+h) - V(x)}{h} \leq \pi M^2.$$

We now take the limit of each of these expressions as $h \to 0$. Since f is continuous, the limit of both m and M as $h \to 0$ is just $f(x)$, while the limit of the central expression is

The argument assumes that $h > 0$, but a similar argument applies for the case $h < 0$.

$$\lim_{h \to 0} \frac{V(x+h) - V(x)}{h} = V'(x),$$

by the definition of the derivative. We therefore have

$$\pi(f(x))^2 \leq V'(x) \leq \pi(f(x))^2,$$

so

$$V'(x) = \pi(f(x))^2.$$

It follows that $V(x)$ is an antiderivative of $\pi(f(x))^2$ and hence that

$$\int_a^b \pi(f(x))^2\, dx = V(b) - V(a).$$

But $V(a) = 0$, so the volume of the solid of revolution generated by the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is

$$V(b) = \pi \int_a^b (f(x))^2\, dx. \tag{4.2}$$

Evaluation of such a volume is therefore equivalent to evaluating the corresponding definite integral. Notice too that if $f(x)$ is non-positive for some values of x in $[a, b]$, then the function $g(x) = |f(x)|$ sweeps out exactly the same solid of revolution as f; see Figure 4.7.

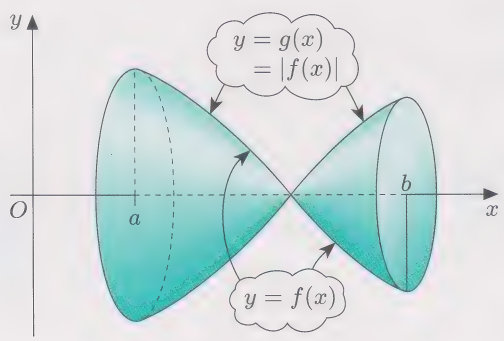

Figure 4.7 Equal volumes

The volume of this solid is

$$\pi \int_a^b (g(x))^2\, dx = \pi \int_a^b |f(x)|^2\, dx = \pi \int_a^b (f(x))^2\, dx,$$

so equation (4.2) holds for *all* functions that are continuous on $[a, b]$, not just for those that are positive on this interval.

In the following result and subsequently, the phrase 'region bounded by the graph of f from $x = a$ to $x = b$' is used as an abbreviation for 'region bounded by the graph of f, the x-axis and the lines $x = a$ and $x = b$'.

Volume of solids of revolution

If f is any function that is continuous on the interval $[a, b]$, then the volume of the solid of revolution generated by the region bounded by the graph of f from $x = a$ to $x = b$ is given by

$$\text{Volume of revolution } = \pi \int_a^b (f(x))^2\, dx. \qquad (4.3)$$

Example 4.1 Finding the volume of a cone

Use integration to find a formula for the volume of a right circular cone of height h and base radius r.

Solution

The cone (on its side) is formed by rotating about the x-axis the shaded triangular region shown in Figure 4.8. The rule for the function f whose graph forms part of the boundary of this region is

$$f(x) = \frac{r}{h}x.$$

So, by equation (4.3), the required volume is

$$\begin{aligned} \pi \int_0^h (f(x))^2\, dx &= \frac{\pi r^2}{h^2} \int_0^h x^2\, dx \\ &= \frac{\pi r^2}{h^2} \left[\tfrac{1}{3}x^3\right]_0^h \\ &= \tfrac{1}{3}\pi r^2 h. \end{aligned}$$

Thus the volume of the cone is $\frac{1}{3}\pi r^2 h$.

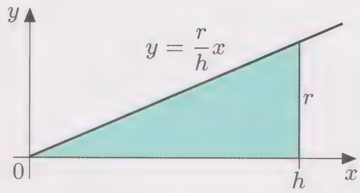

Figure 4.8 A triangular region

The volume of a sphere can be found in a similar way.

Activity 4.1 Volume of a sphere

(a) Show that if r is the radius of the semicircular region in Figure 4.3, then the volume of the solid obtained by rotating the region about the x-axis is given by the integral

$$\pi \int_{-r}^{r} (r^2 - x^2)\, dx.$$

(b) By evaluating this integral, find the volume of a sphere of radius r.

Solutions are given on page 57.

The volumes of solids of revolution in the next three activities are included to provide you with further opportunities to practise your integration techniques.

Activity 4.2 Two more volumes

(a) Find the volume of the solid of revolution obtained when the region bounded by the graph of $f(x) = x^2 + 1$, from $x = 1$ to $x = 2$, is rotated about the x-axis. Give your answer to four decimal places.

(b) Find the volume of the solid of revolution obtained when the region bounded by one arch of the graph of $f(x) = \sin x$ (say from $x = 0$ to $x = \pi$) is rotated about the x-axis. Give your answer to four decimal places.

Solutions are given on page 57.

Activity 4.3 Using integration by parts

(a) Use integration by parts to evaluate the definite integral

$$\int_0^1 xe^{-2x}\,dx.$$

(b) By using integration by parts and your answer to part (a), find the volume of the solid of revolution obtained when the region bounded by the graph of $f(x) = xe^{-x}$, from $x = 0$ to $x = 1$, is rotated about the x-axis. Give your answer to four decimal places.

Solutions are given on page 58.

Activity 4.4 Using integration by substitution

Use integration by substitution to find the volume of the solid of revolution obtained when the region bounded by the graph of $f(x) = x(1 + x^3)$, from $x = 1$ to $x = 3$, is rotated about the x-axis. Give your answer to two decimal places.

A solution is given on page 58.

The remainder of this subsection will not be assessed.

Following the discussion about the meaning of 'area under a graph' in Subsection 1.4, it may have occurred to you to ask

'what is *meant* by the volume of an object with a curved boundary?'

For volumes of solids of revolution, this question can be dealt with in a similar manner to that described earlier for areas. In fact, using the notation introduced in Section 1 and the Fundamental Theorem of Calculus (equation (1.3) on page 20), we may write

$$\pi \int_a^b f^2 = \pi \lim_{N\to\infty} \left(h \sum_{i=0}^{N-1} (f(a+ih))^2 \right), \quad \text{where } h = (b-a)/N.$$

The right-hand side of this equation is a definition of precisely what is meant by the volume of the solid of revolution generated by rotating the region bounded by the graph of a continuous function f between $x = a$ and $x = b$.

For a given number N of steps between a and b, the sum involves N terms each of which is the volume of a cylinder of the form $\pi h(f(a+ih))^2$. Hence this definition of the volume of a solid of revolution involves increasingly close approximation by 'sums of cylinders', of the type shown in Figure 4.9, in the same way that the area under a graph can be described by the limit of successive approximations by 'sums of rectangles'.

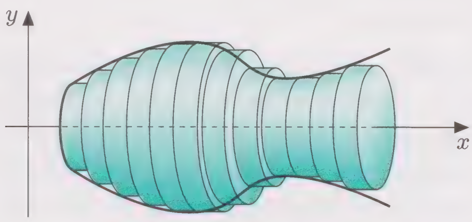

Figure 4.9 Sums of cylinders

Summary of Section 4

This section has introduced:

◇ the idea of generating a solid of revolution by rotating the graph of a continuous function f (along with the region between it and the x-axis) about the x-axis;

◇ the formula for the volume of the solid of revolution generated by the region bounded by the graph of f from $x = a$ to $x = b$, namely

$$\text{Volume of revolution } = \pi \int_a^b (f(x))^2\,dx;$$

◇ examples of volumes of solids of revolution, including the volume $\pi r^2 h$ of a cylinder of radius r and height h, the volume $\frac{1}{3}\pi r^2 h$ of a right circular cone of base radius r and height h, and the volume $\frac{4}{3}\pi r^3$ of a sphere of radius r.

Exercises for Section 4

Exercise 4.1

Find the volume of the solid of revolution obtained when the region bounded by the graph of $f(x) = \sec x$, from $x = 0$ to $x = \frac{1}{4}\pi$, is rotated about the x-axis. Give your answer to four decimal places.

Exercise 4.2

Use integration by substitution to find the volume of the solid of revolution obtained when the region bounded by the graph of $f(x) = \sqrt{x}/(2 + x^2)$, from $x = 0$ to $x = 2$, is rotated about the x-axis. Give your answer to four decimal places.

Exercise 4.3

Use integration by parts to find the volume of the solid of revolution obtained when the region bounded by the graph of $f(x) = xe^{4x}$, from $x = 0$ to $x = 1$, is rotated about the x-axis. Give your answer to two decimal places.

5 Integration with the computer

To study this section you will need access to your computer, together with the Mathcad files for this chapter and Computer Book C.

All the integrals in this chapter can be evaluated using Mathcad.

In this section the use of the computer in obtaining both definite and indefinite integrals is explored.

Computer packages often have limitations where integration is concerned. While they produce the 'right answer' most of the time, you need to be wary of applying them uncritically. Occasionally, as you will see, the numerical answer which a computer package provides for a definite integral can be 'seriously' incorrect, and the algebraic output for an antiderivative may appear in a form which is unhelpful for further analysis.

These limitations do not detract greatly from the usefulness of such packages. The point is that you need to be alert when using them, to check an answer by alternative means where possible and to question results which appear to be at odds with commonsense reasoning.

Refer to Computer Book C for the work in this section.

Summary of Section 5

This section has investigated the use of a computer package for integration and raised issues about the answers which it provides when using it both numerically and symbolically. The moral is that a software package should be used with critical awareness, and that a check on results should be carried out where possible.

Summary of Chapter C2

This chapter has reviewed the process of integration, seen both as the 'undoing' of the process of differentiation and as the limit of a sequence of sums. The relationship between these approaches, which is known as the Fundamental Theorem of Calculus, has been explored.

A table of standard integrals has been given, and the techniques of integration by parts and integration by substitution have been discussed and practised, in particular in obtaining volumes of solids of revolution.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Antiderivative, integral, indefinite integral, arbitrary constant, constant of integration, definite integral from a to b (or over an interval), signed area, integration by parts, integration by substitution, volume of a solid of revolution.

Notation to know and use

$$\int f(x)\,dx, \quad \int f, \quad \int_a^b f(x)\,dx, \quad \int_a^b f, \quad [F(x)]_a^b.$$

Mathematical skills

- ◇ Apply a table of standard integrals.
- ◇ Use a definite integral to evaluate a (signed) area.
- ◇ Use a definite integral to evaluate a volume of a solid of revolution.
- ◇ Use the techniques of 'integration by parts' and 'integration by substitution', expressed symbolically below for indefinite and definite integrals.

Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

Integration by substitution

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \text{ where } u = g(x)$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \text{ where } u = g(x)$$

Mathcad skills

Evaluate integrals using Mathcad, and check the results critically.

Solutions to Activities

Solution 1.1

Each integral is found by reversing a derivative given in Table 1.1, and adding an arbitrary constant c.

(a) $\displaystyle\int \frac{1}{1+u^2}\,du = \arctan u + c$

(b) $\displaystyle\int 3\cos(3x)\,dx = \sin(3x) + c$

(c) $\displaystyle\int 7e^{7t}\,dt = e^{7t} + c$

Solution 1.2

Each integral is found by reversing a derivative given in Table 1.1 and adding an arbitrary constant c. There are many valid choices for the domain. Here the largest interval that contains $-\frac{1}{2}$ and avoids points where the integrand is undefined is used.

(a) Here $(-\infty, 0)$ is the largest interval, containing $-\frac{1}{2}$, on which $1/x$ is defined, so we can write

$$\int \frac{1}{x}\,dx = \ln|x| + c \quad (x < 0).$$

(b) In this case $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ is the largest interval, containing $-\frac{1}{2}$, on which $\sec x \tan x$ is defined, so we can write

$$\int \sec x \tan x\,dx = \sec x + c \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi).$$

(c) Here $(-1, 1)$ is the largest interval, containing $-\frac{1}{2}$, on which $1/\sqrt{1-x^2}$ is defined, so we can write

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c \quad (-1 < x < 1).$$

Solution 1.3

In each case, c is an arbitrary constant.

(a) $\displaystyle\int (5x + x^3)\,dx = \tfrac{5}{2}x^2 + \tfrac{1}{4}x^4 + c$

(b) $$\begin{aligned}\int \frac{1+\sqrt{x}}{x}\,dx &= \int \left(\frac{1}{x} + x^{-1/2}\right)\,dx \\ &= \ln x + 2x^{1/2} + c \quad (x > 0)\end{aligned}$$

(c) $\displaystyle\int (u + 2e^{7u})\,du = \tfrac{1}{2}u^2 + \tfrac{2}{7}e^{7u} + c$

(d) $\displaystyle\int (\sin(2t) - 3\cos(3t))\,dt = -\tfrac{1}{2}\cos(2t) - \sin(3t) + c$

(e) Since $\cos(2x) = 2\cos^2 x - 1$, we have $\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$. So

$$\begin{aligned}\int \cos^2 x\,dx &= \int \tfrac{1}{2}(\cos(2x) + 1)\,dx \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}\sin(2x) + x\right) + c \\ &= \tfrac{1}{4}\sin(2x) + \tfrac{1}{2}x + c.\end{aligned}$$

Solution 1.4

(a) A sketch of the area is shown in Figure S.1

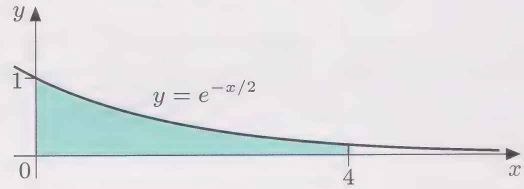

Figure S.1

The area is

$$\begin{aligned}\int_0^4 e^{-x/2}\,dx &= \left[-2e^{-x/2}\right]_0^4 \\ &= -2(e^{-2} - e^0) \\ &= 1.73 \text{ (to 2 d.p.)}.\end{aligned}$$

(b) A sketch of the area is shown in Figure S.2.

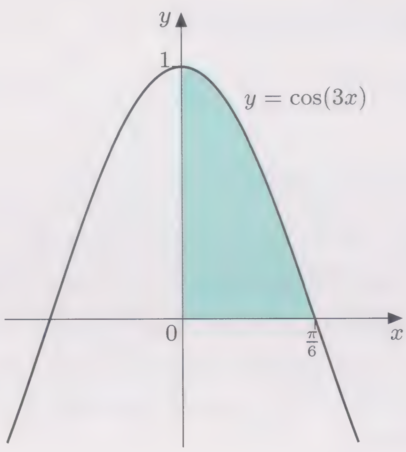

Figure S.2

The area is

$$\begin{aligned}\int_0^{\pi/6} \cos(3x)\,dx &= \left[\tfrac{1}{3}\sin(3x)\right]_0^{\pi/6} \\ &= \tfrac{1}{3}\left(\sin(\tfrac{1}{2}\pi) - \sin 0\right) \\ &= \tfrac{1}{3}.\end{aligned}$$

(c) A sketch of the area is shown in Figure S.3.

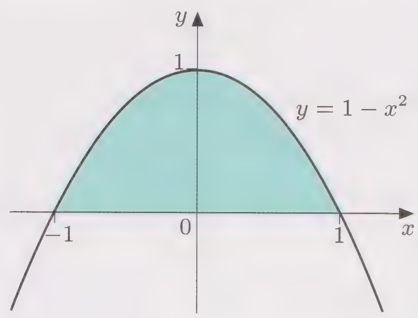

Figure S.3

We have $1 - x^2 = 0$ when $x = \pm 1$, and $1 - x^2 \geq 0$ when $x \in [-1, 1]$, so the required area is given by

$$\begin{aligned}\int_{-1}^{1} (1 - x^2)\, dx &= \left[x - \frac{x^3}{3}\right]_{-1}^{1} \\ &= \left(1 - \tfrac{1}{3}\right) - \left(-1 - \left(-\tfrac{1}{3}\right)\right) \\ &= 2 - \tfrac{2}{3} = \tfrac{4}{3}.\end{aligned}$$

(d) A sketch of the area is shown in Figure S.4.

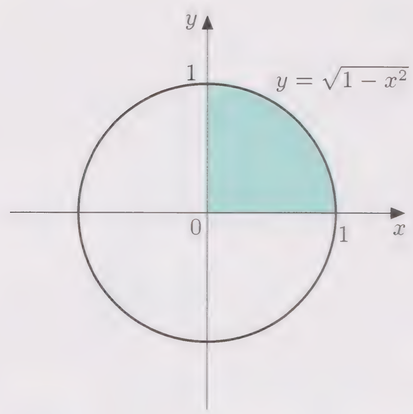

Figure S.4

We have $x^2 + y^2 = 1$, so $y^2 = 1 - x^2$. Taking the (positive) square root (since $y \geq 0$ in the shaded area) gives $y = \sqrt{1 - x^2}$. The graph of $y = \sqrt{1 - x^2}$ cuts the positive x-axis at $x = 1$ and the positive y-axis at $x = 0$, so the required integral is

$$\int_0^1 \sqrt{1 - x^2}\, dx.$$

Solution 1.5

(a) $$\begin{aligned}\int_0^1 \sin(\pi x)\, dx &= -\frac{1}{\pi}\left[\cos(\pi x)\right]_0^1 \\ &= -\frac{1}{\pi}(\cos \pi - \cos 0) \\ &= \frac{2}{\pi} \\ &= 0.64 \text{ (to 2 d.p.)}\end{aligned}$$

(b) $$\begin{aligned}\int_{-1}^{3} (e^{4x} + x^3)\, dx &= \left[\tfrac{1}{4}e^{4x} + \tfrac{1}{4}x^4\right]_{-1}^{3} \\ &= \tfrac{1}{4}(e^{12} + 81 - e^{-4} - 1) \\ &= 20 + \tfrac{1}{4}(e^{12} - e^{-4}) \\ &= 40\,708.69 \text{ (to 2 d.p.)}\end{aligned}$$

(c) $$\begin{aligned}\int_{-\pi/4}^{\pi/4} \sec^2 t\, dt &= \left[\tan t\right]_{-\pi/4}^{\pi/4} \\ &= \tan(\pi/4) - \tan(-\pi/4) \\ &= 2\end{aligned}$$

(d) $$\begin{aligned}\int_{-1}^{1} \frac{1}{1 + u^2}\, du &= \left[\arctan u\right]_{-1}^{1} \\ &= (\pi/4) - (-\pi/4) \\ &= \pi/2 \\ &= 1.57 \text{ (to 2 d.p.)}\end{aligned}$$

(e) $$\begin{aligned}\int_0^1 e^t\, dt &= \left[e^t\right]_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \\ &= 1.72 \text{ (to 2 d.p.)}\end{aligned}$$

(f) $$\begin{aligned}\int_{-2}^{-1} \frac{1}{x}\, dx &= \left[\ln|x|\right]_{-2}^{-1} \\ &= \ln|-1| - \ln|-2| \\ &= \ln(1) - \ln(2) \\ &= 0 - 0.69 \\ &= -0.69 \text{ (to 2 d.p.)}\end{aligned}$$

Solution 1.6

In both cases, let F be any antiderivative of f.

(a) Here we find that

$$\int_a^a f = F(a) - F(a) = 0.$$

(b) In this case we obtain

$$\begin{aligned}\int_a^b f + \int_b^c f &= F(b) - F(a) + F(c) - F(b) \\ &= F(c) - F(a) \\ &= \int_a^c f.\end{aligned}$$

Solution 1.7

Suppose that F is an antiderivative of f, and G is an antiderivative of g. Then

$$(F+G)' = F' + G' = f + g.$$

So $F+G$ is an antiderivative of $f+g$, and we have

$$\begin{aligned}\int_a^b (f+g) &= [F+G]_a^b \\ &= (F(b)+G(b)) - (F(a)+G(a)) \\ &= (F(b)-F(a)) + (G(b)-G(a)) \\ &= \int_a^b f + \int_a^b g.\end{aligned}$$

Solution 1.8

(a) $$\begin{aligned}\int_0^{\pi/2} \cos(3x)\,dx &= \left[\tfrac{1}{3}\sin(3x)\right]_0^{\pi/2} \\ &= \tfrac{1}{3}(-1-0) \\ &= -\tfrac{1}{3}\end{aligned}$$

(b) The area is indicated by the shading in Figure S.5.

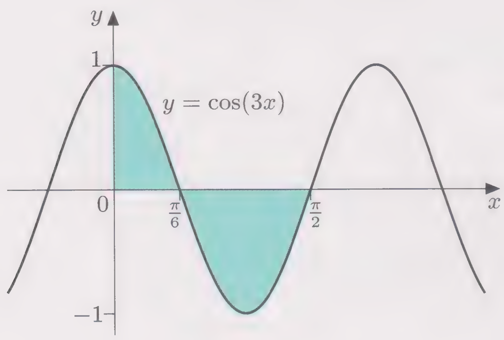

Figure S.5

The graph of $y = \cos(3x)$ is non-negative on $[0, \pi/6]$ and non-positive on $[\pi/6, \pi/2]$.

So the required area is given by

$$A = \int_0^{\pi/6} \cos(3x)\,dx - \int_{\pi/6}^{\pi/2} \cos(3x)\,dx.$$

It follows that

$$\begin{aligned}A &= \left[\tfrac{1}{3}\sin(3x)\right]_0^{\pi/6} - \left[\tfrac{1}{3}\sin(3x)\right]_{\pi/6}^{\pi/2} \\ &= \tfrac{1}{3}(1-0) - \tfrac{1}{3}(-1-1) \\ &= \tfrac{1}{3} + \tfrac{2}{3} \\ &= 1.\end{aligned}$$

Solution 2.1

In each case, the integration by parts formula,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

is used, and c is an arbitrary constant.

(a) Let $f(x) = x$ and $g'(x) = \cos x$; then

$$f'(x) = 1 \text{ and } g(x) = \sin x.$$

Hence

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx \\ &= x\sin x - (-\cos x) + c \\ &= x\sin x + \cos x + c,\end{aligned}$$

as you saw on page 23.

(b) Let $f(x) = x$ and $g'(x) = e^x$; then

$$f'(x) = 1 \text{ and } g(x) = e^x.$$

Hence

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + c \\ &= e^x(x-1) + c.\end{aligned}$$

Solution 2.2

(a) Let $f(x) = \ln x$ and $g'(x) = x$; then

$$f'(x) = 1/x \text{ and } g(x) = x^2/2.$$

Hence

$$\begin{aligned}\int x\ln x\,dx &= \frac{x^2}{2}\ln x - \int \frac{1}{x} \times \frac{x^2}{2}\,dx \\ &= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx \\ &= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + c,\end{aligned}$$

where c is an arbitrary constant.

(b) Using the solution to part (a), we obtain

$$\begin{aligned}\int_1^2 x\ln x\,dx &= \left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^2 \\ &= (2\ln 2 - 1) - \left(\tfrac{1}{2}\ln 1 - \tfrac{1}{4}\right) \\ &= 2\ln 2 - 1 + \tfrac{1}{4} \\ &= 0.6363 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 2.3

In each case, c is an arbitrary constant.

(a) Let $f(x) = x$ and $g'(x) = \cos(2x)$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{2}\sin(2x).$$

Hence

$$\begin{aligned}\int x\cos(2x)\,dx &= \tfrac{1}{2}x\sin(2x) - \int \tfrac{1}{2}\sin(2x)\,dx \\ &= \tfrac{1}{2}x\sin(2x) + \tfrac{1}{4}\cos(2x) + c.\end{aligned}$$

(b) Let $f(x) = x$ and $g'(x) = \sin(3x)$; then

$$f'(x) = 1 \text{ and } g(x) = -\tfrac{1}{3}\cos(3x).$$

Hence

$$\begin{aligned}\int x\sin(3x)\,dx &= -\tfrac{1}{3}x\cos(3x) + \int \tfrac{1}{3}\cos(3x)\,dx\\ &= -\tfrac{1}{3}x\cos(3x) + \tfrac{1}{9}\sin(3x) + c\\ &= \tfrac{1}{9}\sin(3x) - \tfrac{1}{3}x\cos(3x) + c.\end{aligned}$$

Solution 2.4

(a) Let $f(x) = x$ and $g'(x) = e^{2x}$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{2}e^{2x}.$$

Hence

$$\begin{aligned}\int xe^{2x}\,dx &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx\\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + c\\ &= \tfrac{1}{4}e^{2x}(2x-1) + c,\end{aligned}$$

where c is an arbitrary constant.

(b) We have $e^{2x} > 0$ for all values of x, and $x > 0$ if $x \in [1, 2]$, so the product xe^{2x} is positive on this interval. Hence the graph of $y = xe^{2x}$ between $x = 1$ and $x = 2$ lies above the x-axis.

The area under the graph between $x = 1$ and $x = 2$ is therefore given by $\int_1^2 xe^{2x}\,dx$.

(c) The required area is

$$\begin{aligned}\int_1^2 xe^{2x}\,dx &= \left[\tfrac{1}{4}e^{2x}(2x-1)\right]_1^2\\ &= \tfrac{1}{4}e^4(4-1) - \tfrac{1}{4}e^2(2-1)\\ &= \tfrac{1}{4}(3e^4 - e^2)\\ &= 39.1013 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 2.5

Let $f(x) = x$ and $g'(x) = e^{3x}$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{3}e^{3x}.$$

Hence

$$\begin{aligned}\int_0^1 xe^{3x}\,dx &= \left[\tfrac{1}{3}xe^{3x}\right]_0^1 - \tfrac{1}{3}\int_0^1 e^{3x}\,dx\\ &= \tfrac{1}{3}e^3 - \left[\tfrac{1}{9}e^{3x}\right]_0^1\\ &= \tfrac{1}{3}e^3 - \left(\tfrac{1}{9}e^3 - \tfrac{1}{9}e^0\right)\\ &= \tfrac{1}{9}(2e^3 + 1)\\ &= 4.5746 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 2.6

(a) If $x \in [\pi/12, \pi/6]$, then $\theta = 3x \in [\pi/4, \pi/2]$. Now $\sin\theta > 0$ if $\theta \in [\pi/4, \pi/2]$, so we obtain

$$\sin(3x) > 0 \text{ if } x \in [\pi/12, \pi/6].$$

Multiplying a positive value of $\sin(3x)$ by x, which is itself positive on the interval $[\pi/12, \pi/6]$, gives a positive value for the expression $x\sin(3x)$. So the graph of $y = x\sin(3x)$ lies above the x-axis for the whole of the interval $[\pi/12, \pi/6]$.

The area under the graph between $x = \pi/12$ and $x = \pi/6$ is

$$\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx.$$

(b) Let $f(x) = x$ and $g'(x) = \sin(3x)$; then

$$f'(x) = 1 \text{ and } g(x) = -\tfrac{1}{3}\cos(3x).$$

Hence the area is

$$\begin{aligned}&\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx\\ &= \left[-\tfrac{1}{3}x\cos(3x)\right]_{\pi/12}^{\pi/6} + \tfrac{1}{3}\int_{\pi/12}^{\pi/6}\cos(3x)\,dx\\ &= \left(0 + \frac{\pi}{36}\cos\frac{\pi}{4}\right) + \tfrac{1}{9}\left[\sin(3x)\right]_{\pi/12}^{\pi/6}\\ &= \frac{\pi}{36}\times\frac{1}{\sqrt{2}} + \frac{1}{9}\left(1 - \frac{1}{\sqrt{2}}\right)\\ &= 0.0943 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 2.7

In each case, c is an arbitrary constant.

(a) Let $f(x) = \ln x$ and $g'(x) = x^3$; then

$$f'(x) = 1/x \text{ and } g(x) = \tfrac{1}{4}x^4.$$

Hence

$$\begin{aligned}\int x^3\ln x\,dx &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}\int \frac{1}{x}x^4\,dx\\ &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}\int x^3\,dx\\ &= \tfrac{1}{4}\left(x^4\ln x - \frac{x^4}{4}\right) + c\\ &= \tfrac{1}{16}x^4(4\ln x - 1) + c.\end{aligned}$$

(b) Let $f(x) = \ln x$ and $g'(x) = 1$; then

$$f'(x) = 1/x \text{ and } g(x) = x.$$

Hence

$$\begin{aligned}\int \ln x\,dx &= \int \ln x \times 1\,dx\\ &= x\ln x - \int \frac{1}{x}\times x\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + c\\ &= x(\ln x - 1) + c.\end{aligned}$$

Solution 2.8

Let $f(x) = \arctan x$ and $g'(x) = 1$; then

$$f'(x) = \frac{1}{1+x^2} \text{ and } g(x) = x.$$

Hence

$$\begin{aligned}\int \arctan x \, dx &= x \arctan x - \int \frac{1}{1+x^2} \times x \, dx \\ &= x \arctan x - \tfrac{1}{2} \ln(1+x^2) + c,\end{aligned}$$

using the result given.

Solution 2.9

In each case, c is an arbitrary constant.

(a) Let $f(x) = x^2$ and $g'(x) = e^x$; then

$$f'(x) = 2x \text{ and } g(x) = e^x.$$

Hence

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

Using the result of Activity 2.1(b), we have

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - 2e^x(x-1) + c \\ &= e^x(x^2 - 2x + 2) + c.\end{aligned}$$

(b) Let $f(x) = x^2$ and $g'(x) = \cos(3x)$; then

$$f'(x) = 2x \text{ and } g(x) = \tfrac{1}{3}\sin(3x).$$

Hence

$$\int x^2 \cos(3x) \, dx = \tfrac{1}{3} x^2 \sin(3x) - \tfrac{2}{3} \int x \sin(3x) \, dx.$$

Using the result of Activity 2.3(b), we have

$$\begin{aligned}&\int x^2 \cos(3x) \, dx \\ &= \tfrac{1}{3} x^2 \sin(3x) - \tfrac{2}{3} \times \left(\tfrac{1}{9}\sin(3x) - \tfrac{1}{3} x \cos(3x)\right) + c \\ &= \tfrac{1}{27}(9x^2 - 2)\sin(3x) + \tfrac{2}{9} x \cos(3x) + c.\end{aligned}$$

Solution 2.10

In each case, c is an arbitrary constant.

(a) Let $f(x) = \cos(3x)$ and $g'(x) = e^x$; then

$$f'(x) = -3\sin(3x) \text{ and } g(x) = e^x.$$

Hence

$$\int e^x \cos(3x) \, dx = e^x \cos(3x) + 3 \int e^x \sin(3x) \, dx. \tag{S.1}$$

Now use integration by parts again for the remaining integral, setting $f(x)$ to be the trigonometric function.

Let $f(x) = \sin(3x)$ and $g'(x) = e^x$; then

$$f'(x) = 3\cos(3x) \text{ and } g(x) = e^x.$$

Hence

$$\int e^x \sin(3x) \, dx = e^x \sin(3x) - 3 \int e^x \cos(3x) \, dx.$$

Substituting in equation (S.1), we obtain

$$\begin{aligned}&\int e^x \cos(3x) \, dx \\ &= e^x \cos(3x) + 3e^x \sin(3x) - 9 \int e^x \cos(3x) \, dx,\end{aligned}$$

which gives

$$10 \int e^x \cos(3x) \, dx = e^x(\cos(3x) + 3\sin(3x)).$$

Hence

$$\int e^x \cos(3x) \, dx = \tfrac{1}{10} e^x(\cos(3x) + 3\sin(3x)) + c.$$

(b) Let $f(x) = e^{2x}$ and $g'(x) = \sin x$, then

$$f'(x) = 2e^{2x} \text{ and } g(x) = -\cos x.$$

Hence

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx. \tag{S.2}$$

Now use integration by parts again for the remaining integral, setting $f(x)$ to be the exponential function.

Let $f(x) = e^{2x}$ and $g'(x) = \cos x$; then

$$f'(x) = 2e^{2x} \text{ and } g(x) = \sin x.$$

Hence

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx.$$

Substituting in equation (S.2), we obtain

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx,$$

which gives

$$5 \int e^{2x} \sin x \, dx = e^{2x}(2 \sin x - \cos x).$$

Hence

$$\int e^{2x} \sin x \, dx = \tfrac{1}{5} e^{2x}(2 \sin x - \cos x) + c.$$

Solution 3.1

These solutions are presented as a string of equalities, without explicit reference to the steps of the strategy in Frame 2. Also, in each case, c is an arbitrary constant.

(a) Take $u = 5 + 2x^2$; then $du/dx = 4x$. Hence

$$\begin{aligned}\int (5+2x^2)^{16} x\,dx &= \tfrac{1}{4}\int (5+2x^2)^{16}(4x)\,dx \\ &= \tfrac{1}{4}\int u^{16}\,du \\ &= \frac{u^{17}}{4\times 17} + c \\ &= \tfrac{1}{68}(5+2x^2)^{17} + c.\end{aligned}$$

(b) Take $u = x^3$; then $du/dx = 3x^2$. Hence

$$\begin{aligned}\int x^2 \sec^2(x^3)\,dx &= \tfrac{1}{3}\int \sec^2(x^3)(3x^2)\,dx \\ &= \tfrac{1}{3}\int \sec^2 u\,du \\ &= \tfrac{1}{3}\tan u + c \\ &= \tfrac{1}{3}\tan(x^3) + c.\end{aligned}$$

(No domains were asked for in the activity, but if you wanted to choose one here you would have to select one like $\left(-(\pi/2)^{1/3}, (\pi/2)^{1/3}\right)$ that avoids the values of x where the term $\sec^2(x^3)$ in the integrand is undefined.)

(c) Take $u = \cos x$; then $du/dx = -\sin x$. Hence

$$\begin{aligned}\int (\sin x)e^{\cos x}\,dx &= -\int e^{\cos x}(-\sin x)\,dx \\ &= -\int e^u\,du \\ &= -e^u + c \\ &= -e^{\cos x} + c.\end{aligned}$$

(d) Take $u = 3x$; then $du/dx = 3$. Hence

$$\begin{aligned}\int \frac{1}{1+9x^2}\,dx &= \tfrac{1}{3}\int \left(\frac{1}{1+(3x)^2}\right)(3)\,dx \\ &= \tfrac{1}{3}\int \frac{1}{1+u^2}\,du \\ &= \tfrac{1}{3}\arctan u + c \\ &= \tfrac{1}{3}\arctan(3x) + c.\end{aligned}$$

Solution 3.2

Take $u = \cos x$; then $du/dx = -\sin x$. Hence

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\int \frac{1}{\cos x}(-\sin x)\,dx \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= \ln\left(\frac{1}{|u|}\right) + c \\ &= \ln|\sec x| + c,\end{aligned}$$

where c is an arbitrary constant.

Solution 3.3

In each case, c is an arbitrary constant.

(a) Take $u = 6 + x^3$; then $du/dx = 3x^2$. Hence

$$\begin{aligned}\int \left(6+x^3\right)^9 x^2\,dx &= \tfrac{1}{3}\int \left(6+x^3\right)^9 (3x^2)\,dx \\ &= \tfrac{1}{3}\int u^9\,du \\ &= \tfrac{1}{30}u^{10} + c \\ &= \tfrac{1}{30}(6+x^3)^{10} + c.\end{aligned}$$

(b) Take $u = e^{2x}$; then $du/dx = 2e^{2x}$. Hence

$$\begin{aligned}\int e^{2x}\sin(e^{2x})\,dx &= \tfrac{1}{2}\int \sin(e^{2x})(2e^{2x})\,dx \\ &= \tfrac{1}{2}\int \sin u\,du \\ &= -\tfrac{1}{2}\cos u + c \\ &= -\tfrac{1}{2}\cos(e^{2x}) + c.\end{aligned}$$

(c) Take $u = \sin x$; then $du/dx = \cos x$. Hence

$$\begin{aligned}\int \sin^4 x(\cos x)\,dx &= \int u^4\,du \\ &= \frac{u^5}{5} + c \\ &= \tfrac{1}{5}\sin^5 x + c.\end{aligned}$$

(d) Take $u = 1 + x^2$; then $du/dx = 2x$. Hence

$$\begin{aligned}\int x\sqrt{1+x^2}\,dx &= \tfrac{1}{2}\int \sqrt{1+x^2}\,(2x)\,dx \\ &= \tfrac{1}{2}\int \sqrt{u}\,du \\ &= \frac{1}{2}\frac{u^{3/2}}{3/2} + c \\ &= \tfrac{1}{3}(1+x^2)^{3/2} + c.\end{aligned}$$

(e) Take $u = 1 + x^3$; then $du/dx = 3x^2$. Hence

$$\begin{aligned}\int \frac{x^2}{1+x^3}\,dx &= \tfrac{1}{3}\int \frac{1}{1+x^3}(3x^2)\,dx\\ &= \tfrac{1}{3}\int \frac{1}{u}\,du\\ &= \tfrac{1}{3}\ln|u| + c\\ &= \tfrac{1}{3}\ln|1+x^3| + c.\end{aligned}$$

Solution 3.4

(a) Take $u = 1 + 2x^2$; then $du/dx = 4x$. Also we have $u = 1$ when $x = 0$, and $u = 3$ when $x = 1$. Hence

$$\begin{aligned}\int_0^1 \frac{x}{1+2x^2}\,dx &= \tfrac{1}{4}\int_0^1 \left(\frac{1}{1+2x^2}\right)(4x)\,dx\\ &= \tfrac{1}{4}\int_1^3 \frac{1}{u}\,du\\ &= \tfrac{1}{4}\left[\ln|u|\right]_1^3\\ &= \tfrac{1}{4}(\ln 3 - \ln 1)\\ &= \tfrac{1}{4}\ln 3\\ &= 0.2747 \text{ (to 4 d.p.).}\end{aligned}$$

(b) Take $u = 1 + \cos^2 x$; then $du/dx = -2\cos x \sin x$. Also we have $u = 2$ when $x = 0$, and $u = 1$ when $x = \pi/2$. Hence

$$\begin{aligned}&\int_0^{\pi/2} \frac{\sin x \cos x}{1+\cos^2 x}\,dx\\ &= -\tfrac{1}{2}\int_0^{\pi/2}\left(\frac{1}{1+\cos^2 x}\right)(-2\cos x \sin x)\,dx\\ &= -\tfrac{1}{2}\int_2^1 \frac{1}{u}\,du\\ &= -\tfrac{1}{2}[\ln|u|]_2^1\\ &= \tfrac{1}{2}\ln 2\\ &= 0.3466 \text{ (to 4 d.p.).}\end{aligned}$$

(c) Take $u = -3x^2 - 2$; then $du/dx = -6x$. Also we have $u = -5$ when $x = -1$, and $u = -2$ when $x = 0$. Hence

$$\begin{aligned}\int_{-1}^0 xe^{-3x^2-2}\,dx &= -\tfrac{1}{6}\int_{-1}^0 \left(e^{-3x^2-2}\right)(-6x)\,dx\\ &= -\tfrac{1}{6}\int_{-5}^{-2} e^u\,du\\ &= -\tfrac{1}{6}\left[e^u\right]_{-5}^{-2}\\ &= \tfrac{1}{6}\left(e^{-5} - e^{-2}\right)\\ &= -0.0214 \text{ (to 4 d.p.).}\end{aligned}$$

(d) Take $u = \sec x$; then $du/dx = \sec x \tan x$. Also we have $u = 1$ when $x = 0$, and $u = \sqrt{2}$ when $x = \pi/4$. Hence

$$\begin{aligned}\int_0^{\pi/4} \sec^3 x \tan x\,dx &= \int_0^{\pi/4} \sec^2 x(\sec x \tan x)\,dx\\ &= \int_1^{\sqrt{2}} u^2\,du\\ &= \left[\frac{u^3}{3}\right]_1^{\sqrt{2}}\\ &= \tfrac{1}{3}(2\sqrt{2} - 1)\\ &= 0.6095 \text{ (to 4 d.p.).}\end{aligned}$$

Solution 3.5

In each case, c is an arbitrary constant.

(a) Take $u = 1 + 2x$; then $du/dx = 2$ and $x = \frac{1}{2}(u-1)$. Hence

$$\begin{aligned}\int \frac{x}{(1+2x)^3}\,dx &= \tfrac{1}{2}\int \frac{x}{(1+2x)^3}(2)\,dx\\ &= \tfrac{1}{4}\int \frac{u-1}{u^3}\,du\\ &= \tfrac{1}{4}\int (u^{-2} - u^{-3})\,du\\ &= \tfrac{1}{4}\left(\frac{u^{-1}}{-1} - \frac{u^{-2}}{-2}\right) + c\\ &= \tfrac{1}{8}\left(\frac{1}{u^2} - \frac{2}{u}\right) + c\\ &= \tfrac{1}{8}\left(\frac{1}{(1+2x)^2} - \frac{2}{1+2x}\right) + c.\end{aligned}$$

(b) Take $u = 1 + x$; then $du/dx = 1$ and $x = u - 1$. Hence

$$\begin{aligned}\int x\sqrt{1+x}\,dx &= \int (u-1)\sqrt{u}\,du\\ &= \int \left(u^{3/2} - u^{1/2}\right)du\\ &= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + c\\ &= \tfrac{2}{5}(1+x)^{5/2} - \tfrac{2}{3}(1+x)^{3/2} + c.\end{aligned}$$

(c) Take $u = x + 2$; then $du/dx = 1$ and $x = u - 2$, so that $2x + 3 = 2u - 1$. Hence

$$\begin{aligned}\int \frac{2x+3}{(x+2)^{1/3}}\,dx &= \int \frac{2u-1}{u^{1/3}}\,du\\ &= \int \left(2u^{2/3} - u^{-1/3}\right)du\\ &= \frac{2u^{5/3}}{5/3} - \frac{u^{2/3}}{2/3} + c\\ &= \tfrac{6}{5}(x+2)^{5/3} - \tfrac{3}{2}(x+2)^{2/3} + c.\end{aligned}$$

Solution 3.6

Take $x = \tan u$, where $u = \arctan x$. Then $dx/du = \sec^2 u$. Hence

$$\begin{aligned}\int \frac{1}{(1+x^2)^2}\,dx &= \int \frac{1}{(1+\tan^2 u)^2}(\sec^2 u)\,du \\ &= \int \frac{1}{(\sec^2 u)^2}(\sec^2 u)\,du \\ &= \int \cos^2 u\,du \\ &= \int \tfrac{1}{2}(1+\cos(2u))\,du \\ &= \tfrac{1}{2}u + \tfrac{1}{4}\sin(2u) + c \\ &= \tfrac{1}{2}\arctan x + \tfrac{1}{4}\sin(2\arctan x) + c,\end{aligned}$$

where c is an arbitrary constant.

Solution 3.7

Take $x = (u-1)^2$, where $u = 1+\sqrt{x}$. Then $dx/du = 2(u-1)$. Since $u = 1+\sqrt{x}$, we have $u = 2$ when $x = 1$, and $u = 3$ when $x = 4$. Hence

$$\begin{aligned}\int_1^4 \frac{x}{1+\sqrt{x}}\,dx &= \int_2^3 \frac{(u-1)^2}{u}(2(u-1))\,du \\ &= \int_2^3 \frac{2(u-1)^3}{u}\,du \\ &= \int_2^3 \left(2u^2 - 6u + 6 - \frac{2}{u}\right)du \\ &= \left[\tfrac{2}{3}u^3 - 3u^2 + 6u - 2\ln|u|\right]_2^3 \\ &= (18 - 27 + 18 - 2\ln 3) \\ &\quad - (\tfrac{16}{3} - 12 + 12 - 2\ln 2) \\ &= \tfrac{11}{3} + 2\ln(\tfrac{2}{3}) \\ &= 2.8557 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 3.8

(a) Since

$$\frac{d}{dx}(5x^4) = 20x^3,$$

and x^3 is a multiple of this, we can try the substitution $u = 5x^4$.

(b) The integrand is a product, and the term x will give a constant when differentiated, which suggests use of integration by parts (compare with Activity 2.3).

(c) This is already a sum of standard functions.

(d) The integrand is very similar to $1/(1+x^2)$, whose integral is $\arctan x$. We could turn the integrand into that form by substituting $u = \sqrt{2}x$, so try that (compare with Activity 3.1(d)).

(e) Since

$$\frac{d}{dx}(7-x^3) = -3x^2,$$

and x^2 is a multiple of this, try the substitution $u = 7 - x^3$.

(One possibility for a domain of the integral is $(7^{\frac{1}{3}}, \infty)$, another possibility is $(-\infty, 7^{\frac{1}{3}})$.)

(f) The integrand is a product, and the term x will give a constant when differentiated, which again suggests use of integration by parts (compare with Activity 2.4(a)).

(g) This can be integrated by treating '$x-1$' as if it were 'x', since the derivative of $x-1$ is 1. Thus an antiderivative of $(x-1)^4$ is $(x-1)^5/5$.

Using the substitution $u = x-1$ is a more routine method.

Solution 4.1

(a) The shaded region on the left in Figure 4.3 is bounded above by a semicircle, on which (since r is the radius) $x^2+y^2=r^2$, with $y \geq 0$. From this we obtain $y = \sqrt{r^2-x^2}$, where $x \in [-r, r]$.

Rotating this region about the x-axis produces a sphere, with volume given by

$$\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx = \pi\int_{-r}^{r}\left(r^2-x^2\right)dx.$$

(b) The volume is

$$\begin{aligned}\pi\int_{-r}^{r}\left(r^2-x^2\right)dx &= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} \\ &= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right) \\ &= \pi\left(2r^3 - \frac{2r^3}{3}\right) = \frac{4}{3}\pi r^3.\end{aligned}$$

Thus the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution 4.2

(a) The volume is

$$\begin{aligned}\pi\int_1^2\left(x^2+1\right)^2 dx &= \pi\int_1^2\left(x^4+2x^2+1\right)dx \\ &= \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_1^2 \\ &= \pi\left((\tfrac{32}{5} + \tfrac{16}{3} + 2)\right. \\ &\quad \left.- (\tfrac{1}{5} + \tfrac{2}{3} + 1)\right) \\ &= \tfrac{178}{15}\pi \\ &= 37.2802 \text{ (to 4 d.p.)}.\end{aligned}$$

(b) The volume is

$$\begin{aligned}\pi\int_0^{\pi}\sin^2 x\,dx &= \frac{\pi}{2}\int_0^{\pi}(1-\cos(2x))\,dx\\ &= \frac{\pi}{2}\left[x-\tfrac{1}{2}\sin(2x)\right]_0^{\pi}\\ &= \frac{\pi}{2}\big(\pi-\tfrac{1}{2}\sin(2\pi)\\ &\qquad -\left(0-\tfrac{1}{2}\sin 0\right)\big)\\ &= \frac{\pi^2}{2}\\ &= 4.9348 \text{ (to 4 d.p.).}\end{aligned}$$

Solution 4.3

(a) Let $f(x)=x$ and $g'(x)=e^{-2x}$; then

$$f'(x)=1 \text{ and } g(x)=-\tfrac{1}{2}e^{-2x}.$$

Hence

$$\begin{aligned}\int_0^1 xe^{-2x}\,dx &= \left[-\tfrac{1}{2}xe^{-2x}\right]_0^1+\tfrac{1}{2}\int_0^1 e^{-2x}\,dx\\ &= \left(-\tfrac{1}{2}e^{-2}+0\right)+\tfrac{1}{2}\left[-\tfrac{1}{2}e^{-2x}\right]_0^1\\ &= -\tfrac{1}{2}e^{-2}-\tfrac{1}{4}\left(e^{-2}-e^{0}\right)\\ &= -\tfrac{3}{4}e^{-2}+\tfrac{1}{4}\\ &= \tfrac{1}{4}(1-3e^{-2})\\ &= 0.1485 \text{ (to 4 d.p.).}\end{aligned}$$

(b) The required volume is

$$\pi\int_0^1 (xe^{-x})^2\,dx = \pi\int_0^1 x^2e^{-2x}\,dx.$$

We use integration by parts.
Let $f(x)=x^2$ and $g'(x)=e^{-2x}$; then

$$f'(x)=2x \text{ and } g(x)=-\tfrac{1}{2}e^{-2x}.$$

Hence

$$\begin{aligned}\pi\int_0^1 x^2e^{-2x}\,dx &= \pi\left[-\tfrac{1}{2}x^2e^{-2x}\right]_0^1\\ &\qquad -\pi\int_0^1(-xe^{-2x})\,dx\\ &= -\frac{\pi}{2}\left(e^{-2}-0\right)\\ &\qquad +\frac{\pi}{4}\left(1-3e^{-2}\right) \text{ (from part (a))}\\ &= \frac{\pi}{4}(1-5e^{-2})\\ &= 0.2539 \text{ (to 4 d.p.).}\end{aligned}$$

Solution 4.4

The volume is $\pi\displaystyle\int_1^3 x^2(1+x^3)^2\,dx$.

Let $u=1+x^3$; then $\dfrac{du}{dx}=3x^2$. Also $u=2$ when $x=1$, and $u=28$ when $x=3$. Hence

$$\begin{aligned}\pi\int_1^3 x^2(1+x^3)^2\,dx &= \frac{\pi}{3}\int_1^3(1+x^3)^2(3x^2)\,dx\\ &= \frac{\pi}{3}\int_2^{28}u^2\,du\\ &= \frac{\pi}{3}\left[\frac{u^3}{3}\right]_2^{28}\\ &= \frac{\pi}{9}(28^3-2^3)\\ &= 7659.90 \text{ (to 2 d.p.).}\end{aligned}$$

Solutions to Exercises

Solution 1.1

In each case, c is an arbitrary constant.

(a) $\int (x^{18} + x)\,dx = \dfrac{x^{19}}{19} + \dfrac{x^2}{2} + c$

(b) $\displaystyle\int \frac{-6}{x^4}\,dx = \frac{2}{x^3} + c$

(c) $\int \sin\left(\tfrac{1}{3}x\right)\,dx = -3\cos\left(\tfrac{1}{3}x\right) + c$

(d) $\int e^{-2x}\,dx = -\tfrac{1}{2}e^{-2x} + c$

(e) $\int (\sec^2 x + \operatorname{cosec}^2 x)\,dx = \tan x - \cot x + c$

(f) We have

$$\begin{aligned}&\frac{(3x^2+1)(x-1)}{\sqrt{x}}\\ &= \frac{3x^3 - 3x^2 + x - 1}{\sqrt{x}}\\ &= 3x^{5/2} - 3x^{3/2} + x^{1/2} - x^{-1/2}.\end{aligned}$$

So

$$\begin{aligned}&\int \frac{(3x^2+1)(x-1)}{\sqrt{x}}\,dx\\ &= \int \left(3x^{5/2} - 3x^{3/2} + x^{1/2} - x^{-1/2}\right)dx\\ &= \tfrac{6}{7}x^{7/2} - \tfrac{6}{5}x^{5/2} + \tfrac{2}{3}x^{3/2} - 2x^{1/2} + c.\end{aligned}$$

Solution 1.2

(a) $\int_0^1 7x^8\,dx = \tfrac{7}{9}\left[x^9\right]_0^1 = \tfrac{7}{9}(1-0) = \tfrac{7}{9}$

(b) $\int_0^{\pi/4} \cos(2x)\,dx = \tfrac{1}{2}\left[\sin(2x)\right]_0^{\pi/4} = \tfrac{1}{2}(1-0) = \tfrac{1}{2}$

(c) $$\begin{aligned}\int_{-1}^{1} e^{7x}\,dx &= \tfrac{1}{7}\left[e^{7x}\right]_{-1}^{1} = \tfrac{1}{7}\left(e^7 - e^{-7}\right)\\ &= 156.66 \text{ (to 2 d.p.)}\end{aligned}$$

(d) $$\begin{aligned}\int_{\pi/4}^{\pi/3} \operatorname{cosec}^2 x\,dx &= -\left[\cot x\right]_{\pi/4}^{\pi/3}\\ &= -\left(\cot(\pi/3) - \cot(\pi/4)\right)\\ &= -\left(\frac{1}{\sqrt{3}} - 1\right)\\ &= 1 - \frac{1}{\sqrt{3}} = 0.42 \text{ (to 2 d.p.)}\end{aligned}$$

(e) Since $\cos(2x) = 1 - 2\sin^2 x$, we have

$$\sin^2 x = \tfrac{1}{2}(1 - \cos(2x)).$$

So

$$\begin{aligned}\int_0^{\pi/4} \sin^2 x\,dx &= \tfrac{1}{2}\int_0^{\pi/4} (1 - \cos(2x))\,dx\\ &= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin(2x)\right]_0^{\pi/4}\\ &= \tfrac{1}{2}\left((\pi/4) - \tfrac{1}{2}\right)\\ &= \pi/8 - \tfrac{1}{4} = 0.14 \text{ (to 2 d.p.)}.\end{aligned}$$

Solution 1.3

(a) The graph lies above the x-axis for $x \in [-1, 0]$, so the area is

$$\begin{aligned}\int_{-1}^{0} \left(x^2 - 4x\right)\,dx &= \left[\frac{x^3}{3} - 2x^2\right]_{-1}^{0}\\ &= (0-0) - \left(-\tfrac{1}{3} - 2\right)\\ &= \tfrac{7}{3}.\end{aligned}$$

(b) The graph lies below the x-axis for $x \in [0, 2]$, so the area is

$$\begin{aligned}-\int_0^2 \left(x^2 - 4x\right)\,dx &= -\left[\frac{x^3}{3} - 2x^2\right]_0^2\\ &= -\left(\tfrac{8}{3} - 8\right) + (0-0)\\ &= \tfrac{16}{3}.\end{aligned}$$

(c) $$\begin{aligned}\int_{-1}^{2} (x^2 - 4x)\,dx &= \left[\frac{x^3}{3} - 2x^2\right]_{-1}^{2}\\ &= \left(\tfrac{8}{3} - 8\right) - \left(\tfrac{-1}{3} - 2\right)\\ &= -\tfrac{16}{3} + \tfrac{7}{3}\\ &= -3\end{aligned}$$

(d) Definite integrals represent areas only when the graph concerned is non-negative throughout the interval of integration, which is not so here. In this case the total area is equal to $\tfrac{7}{3} + \tfrac{16}{3} = \tfrac{23}{3}$, whereas the definite integral (*signed* area) is equal to

$$\begin{aligned}&\int_{-1}^{0} (x^2 - 4x)\,dx + \int_0^2 (x^2 - 4x)\,dx\\ &= \tfrac{7}{3} - \tfrac{16}{3}\\ &= -3.\end{aligned}$$

Solution 2.1

In each of parts (a)–(e), c is arbitrary constant.

(a) Let $f(x) = x$ and $g'(x) = e^{-x}$; then

$$f'(x) = 1 \text{ and } g(x) = -e^{-x}.$$

Substituting into the integration by parts formula, we obtain

$$\begin{aligned}\int xe^{-x}\,dx &= -xe^{-x} - \int (-e^{-x})\,dx\\ &= -xe^{-x} - e^{-x} + c\\ &= -e^{-x}(x+1) + c.\end{aligned}$$

(b) Let $f(x) = \ln x$ and $g'(x) = \sqrt{x} = x^{1/2}$; then

$$f'(x) = 1/x \text{ and } g(x) = \tfrac{2}{3}x^{3/2}.$$

Substituting into the integration by parts formula, we obtain

$$\begin{aligned}\int \sqrt{x}\ln x\,dx &= (\ln x)\tfrac{2}{3}x^{3/2} - \int (1/x)\times \tfrac{2}{3}x^{3/2}\,dx\\ &= \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx\\ &= \tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + c\\ &= \tfrac{2}{9}x^{3/2}(3\ln x - 2) + c.\end{aligned}$$

(c) Let $f(x) = x$ and $g'(x) = \cos(4x)$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{4}\sin(4x).$$

Hence

$$\begin{aligned}\int x\cos(4x)\,dx &= \tfrac{1}{4}x\sin(4x) - \tfrac{1}{4}\int \sin(4x)\,dx\\ &= \tfrac{1}{4}x\sin(4x) + \tfrac{1}{16}\cos(4x) + c.\end{aligned}$$

(d) Let $f(x) = x^2$ and $g'(x) = \sin(3x)$; then

$$f'(x) = 2x \text{ and } g(x) = -\tfrac{1}{3}\cos(3x).$$

Hence

$$\begin{aligned}&\int x^2\sin(3x)\,dx\\ &= -\tfrac{1}{3}x^2\cos(3x) + \tfrac{2}{3}\int x\cos(3x)\,dx. \qquad \text{(S.3)}\end{aligned}$$

To evaluate the remaining integral, we use integration by parts once more.

Let $f(x) = x$ and $g'(x) = \cos(3x)$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{3}\sin(3x).$$

Hence

$$\begin{aligned}\int x\cos(3x)\,dx &= \tfrac{1}{3}x\sin(3x) - \tfrac{1}{3}\int \sin(3x)\,dx\\ &= \tfrac{1}{3}x\sin(3x) + \tfrac{1}{9}\cos(3x).\end{aligned}$$

Using this result to substitute for the integral on the right-hand side of equation (S.3), we obtain

$$\begin{aligned}&\int x^2\sin(3x)\,dx\\ &= -\tfrac{1}{3}x^2\cos(3x) + \tfrac{2}{3}\left(\tfrac{1}{3}x\sin(3x) + \tfrac{1}{9}\cos(3x)\right) + c\\ &= \left(\tfrac{2}{27} - \tfrac{1}{3}x^2\right)\cos(3x) + \tfrac{2}{9}x\sin(3x) + c.\end{aligned}$$

(e) Let $f(x) = e^{3x}$ and $g'(x) = \cos x$; then

$$f'(x) = 3e^{3x} \text{ and } g(x) = \sin x.$$

Hence

$$\begin{aligned}\int e^{3x}\cos x\,dx &= e^{3x}\sin x - \int 3e^{3x}\sin x\,dx\\ &= e^{3x}\sin x - 3\int e^{3x}\sin x\,dx. \qquad \text{(S.4)}\end{aligned}$$

To evaluate the remaining integral, we use integration by parts once more.

Let $f(x) = e^{3x}$ and $g'(x) = \sin x$; then

$$f'(x) = 3e^{3x} \text{ and } g(x) = -\cos x.$$

Hence

$$\begin{aligned}\int e^{3x}\sin x\,dx &= e^{3x}(-\cos x) - \int 3e^{3x}(-\cos x)\,dx\\ &= -e^{3x}\cos x + 3\int e^{3x}\cos x\,dx.\end{aligned}$$

Substituting into equation (S.4), we obtain

$$\begin{aligned}&\int e^{3x}\cos x\,dx\\ &= e^{3x}\sin x - 3\left(-e^{3x}\cos x + 3\int e^{3x}\cos x\,dx\right)\\ &= e^{3x}\sin x + 3e^{3x}\cos x - 9\int e^{3x}\cos x\,dx,\end{aligned}$$

which gives

$$10\int e^{3x}\cos x\,dx = e^{3x}(\sin x + 3\cos x).$$

Hence

$$\int e^{3x}\cos x\,dx = \tfrac{1}{10}e^{3x}(\sin x + 3\cos x) + c.$$

(f) Using the result of part (a), we have

$$\begin{aligned}\int_0^1 xe^{-x}\,dx &= \left[-e^{-x}(x+1)\right]_0^1\\ &= -2e^{-1} + e^0\\ &= 1 - 2e^{-1}\\ &= 0.2642 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 3.1

In each of parts (a), (c) and (d), c is an arbitrary constant.

(a) Take $u = 1 + x^2$; then $du/dx = 2x$. Hence

$$\begin{aligned}\int x(1+x^2)^3\,dx &= \tfrac{1}{2}\int (1+x^2)^3(2x)\,dx\\ &= \tfrac{1}{2}\int u^3\,du\\ &= \tfrac{1}{8}u^4 + c\\ &= \tfrac{1}{8}(1+x^2)^4 + c.\end{aligned}$$

(b) Take $u = \cos x$; then $du/dx = -\sin x$. Also $u = 1$ when $x = 0$, and $u = 1/\sqrt{2}$ when $x = \pi/4$. Hence

$$\begin{aligned}\int_0^{\pi/4} \sin x\cos^3 x\,dx &= -\int_0^{\pi/4} \cos^3 x\,(-\sin x)\,dx\\ &= -\int_1^{1/\sqrt{2}} u^3\,du\\ &= -\left[\frac{u^4}{4}\right]_1^{1/\sqrt{2}}\\ &= -\left(\tfrac{1}{16} - \tfrac{1}{4}\right)\\ &= \tfrac{3}{16}.\end{aligned}$$

(c) Take $u = x^2 - 4x + 3$; then $du/dx = 2x - 4$. Hence

$$\begin{aligned}\int \frac{x-2}{x^2-4x+3}\,dx &= \tfrac{1}{2}\int \frac{1}{x^2-4x+3}(2x-4)\,dx\\ &= \tfrac{1}{2}\int \frac{1}{u}\,du\\ &= \tfrac{1}{2}\ln|u| + c\\ &= \tfrac{1}{2}\ln|x^2-4x+3| + c.\end{aligned}$$

(d) Take $u = 1 + x^3$; then $du/dx = 3x^2$. Also $x^3 = u - 1$. Hence

$$\begin{aligned}\int \frac{x^5}{1+x^3}\,dx &= \tfrac{1}{3}\int \frac{x^3}{1+x^3}(3x^2)\,dx\\ &= \tfrac{1}{3}\int \frac{u-1}{u}\,du\\ &= \tfrac{1}{3}\int \left(1-\frac{1}{u}\right)du\\ &= \tfrac{1}{3}u - \tfrac{1}{3}\ln|u| + c\\ &= \tfrac{1}{3}(1+x^3) - \tfrac{1}{3}\ln|1+x^3| + c.\end{aligned}$$

(e) Take $u = \sqrt{x} - 1 = x^{1/2} - 1$; then $du/dx = \frac{1}{2}x^{-1/2}$. Also $u = 1$ when $x = 4$, and $u = 2$ when $x = 9$. Hence

$$\begin{aligned}\int_4^9 \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx &= 2\int_4^9 \frac{1}{\sqrt{x}-1}\left(\tfrac{1}{2}x^{-1/2}\right)dx\\ &= 2\int_1^2 \frac{1}{u}\,du\\ &= 2\left[\ln|u|\right]_1^2\\ &= 2\ln 2.\end{aligned}$$

Solution 3.2

In each of parts (b) and (c), c is an arbitrary constant.

(a) Take $u = 1 + x^2$; then $du/dx = 2x$. Also $u = 1$ when $x = 0$, and $u = 2$ when $x = 1$. Hence

$$\begin{aligned}\int_0^1 x(1+x^2)^5\,dx &= \tfrac{1}{2}\int_0^1 (1+x^2)^5(2x)\,dx\\ &= \tfrac{1}{2}\int_1^2 u^5\,du\\ &= \tfrac{1}{12}\left[u^6\right]_1^2\\ &= \tfrac{1}{12}(64-1) = \tfrac{21}{4}.\end{aligned}$$

(b) Take $u = \sin x$; then $du/dx = \cos x$. Hence

$$\begin{aligned}\int \cos x \sin^3 x\,dx &= \int \sin^3 x\,(\cos x)\,dx\\ &= \int u^3\,du\\ &= \tfrac{1}{4}u^4 + c\\ &= \tfrac{1}{4}\sin^4 x + c.\end{aligned}$$

(c) Take $u = 4x^2 + 3x + 5$; then $du/dx = 8x + 3$. Hence

$$\begin{aligned}&\int (24x+9)(4x^2+3x+5)^7\,dx\\ &\quad= 3\int (4x^2+3x+5)^7(8x+3)\,dx\\ &\quad= 3\int u^7\,du\\ &\quad= \tfrac{3}{8}u^8 + c\\ &\quad= \tfrac{3}{8}(4x^2+3x+5)^8 + c.\end{aligned}$$

(d) Take $u = 3x^2 + 4x + 16$; then $du/dx = 6x + 4$. Also $u = 16$ when $x = 0$, and $u = 36$ when $x = 2$. Hence

$$\begin{aligned}&\int_0^2 \frac{6x+4}{\sqrt{3x^2+4x+16}}\,dx\\ &\quad= \int_0^2 \frac{1}{\sqrt{3x^2+4x+16}}(6x+4)\,dx\\ &\quad= \int_{16}^{36} u^{-1/2}\,du\\ &\quad= \left[2u^{1/2}\right]_{16}^{36}\\ &\quad= 4.\end{aligned}$$

Solution 3.3

In each of parts (a) and (c), c is an arbitrary constant.

(a) Take $x = u^2 + 2$, where $u = \sqrt{x-2} = (x-2)^{1/2}$. Then $dx/du = 2u$. Hence

$$\begin{aligned}\int \frac{x}{\sqrt{x-2}}\,dx &= \int \frac{u^2+2}{u}(2u)\,du\\ &= \int (2u^2+4)\,du\\ &= \frac{2u^3}{3} + 4u + c\\ &= \tfrac{2}{3}(x-2)^{3/2} + 4(x-2)^{1/2} + c.\end{aligned}$$

(b) Take $x = 3\tan u$, where $u = \arctan(x/3)$. Then $dx/du = 3\sec^2 u$. Since $u = \arctan(x/3)$, we have $u = 0$ when $x = 0$, and $u = \frac{1}{4}\pi$ when $x = 3$. Hence

$$\begin{aligned}\int_0^3 \frac{1}{9+x^2}\,dx &= \int_0^{\pi/4}\left(\frac{1}{9+(3\tan u)^2}\right)(3\sec^2 u)\,du\\ &= \tfrac{1}{3}\int_0^{\pi/4}\left(\frac{\sec^2 u}{1+\tan^2 u}\right)du\\ &= \tfrac{1}{3}\int_0^{\pi/4} 1\,du\\ &= \tfrac{1}{3}[u]_0^{\pi/4}\\ &= \tfrac{1}{12}\pi.\end{aligned}$$

(c) Take $x = \sin u$, where $u = \arcsin x$. Then $dx/du = \cos u$. Hence

$$\begin{aligned}\int (1-x^2)^{-3/2}\,dx &= \int (1-\sin^2 u)^{-3/2}(\cos u)\,du\\ &= \int (\cos^2 u)^{-3/2}(\cos u)\,du\\ &= \int \sec^2 u\,du\\ &= \tan u + c\\ &= \tan(\arcsin x) + c.\end{aligned}$$

Solution 3.4

In each case, c is an arbitrary constant.

(a) Take $u = \tan x$; then $du/dx = \sec^2 x$. Hence

$$\begin{aligned}\int \frac{\sec^2 x}{\tan x}\,dx &= \int \frac{1}{\tan x}(\sec^2 x)\,dx\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + c\\ &= \ln|\tan x| + c.\end{aligned}$$

(b) Take $u = 1 + x^2$; then $du/dx = 2x$ and $x^2 = u - 1$. Hence

$$\begin{aligned}\int \frac{x^3}{(1+x^2)^{1/2}}\,dx &= \tfrac{1}{2}\int \frac{x^2}{(1+x^2)^{1/2}}(2x)\,dx\\ &= \tfrac{1}{2}\int \frac{u-1}{u^{1/2}}\,du\\ &= \tfrac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du\\ &= \tfrac{1}{3}u^{3/2} - u^{1/2} + c\\ &= \tfrac{1}{3}\left(1+x^2\right)^{3/2} - \left(1+x^2\right)^{1/2} + c.\end{aligned}$$

(c) Take $u = e^x$; then $du/dx = e^x$. Hence

$$\begin{aligned}\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx &= \int \frac{1}{\sqrt{1-e^{2x}}}(e^x)\,dx\\ &= \int \frac{1}{\sqrt{1-u^2}}\,du\\ &= \arcsin u + c\\ &= \arcsin(e^x) + c.\end{aligned}$$

Solution 3.5

The choice of method was discussed in Activity 3.8.

In each case, c is an arbitrary constant.

(a) Here we use integration by substitution.

Take $u = 5x^4$; then $du/dx = 20x^3$. Hence

$$\begin{aligned}\int x^3\cos(5x^4)\,dx &= \tfrac{1}{20}\int \cos(5x^4)(20x^3)\,dx\\ &= \tfrac{1}{20}\int \cos u\,du\\ &= \tfrac{1}{20}\sin u + c\\ &= \tfrac{1}{20}\sin(5x^4) + c.\end{aligned}$$

(b) Here we use integration by parts.

Let $f(x) = x$ and $g'(x) = \cos(5x)$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{5}\sin(5x).$$

Hence

$$\begin{aligned}\int x\cos(5x)\,dx &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\int \sin(5x)\,dx\\ &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\left(-\tfrac{1}{5}\cos(5x)\right) + c\\ &= \tfrac{1}{5}x\sin(5x) + \tfrac{1}{25}\cos(5x) + c.\end{aligned}$$

(c) Here neither substitution nor parts is required:

$$\int (x + \cos(5x))\,dx = \tfrac{1}{2}x^2 + \tfrac{1}{5}\sin(5x) + c.$$

(d) Here we use integration by substitution. Take $u = \sqrt{2}x$; then $du/dx = \sqrt{2}$. Hence

$$\begin{aligned}\int \frac{1}{1+2x^2}\,dx &= \frac{1}{\sqrt{2}}\int \frac{1}{1+(\sqrt{2}x)^2}\left(\sqrt{2}\right)dx\\ &= \frac{1}{\sqrt{2}}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{\sqrt{2}}\arctan u + c\\ &= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}x\right) + c.\end{aligned}$$

(e) Here we use integration by substitution.

Take $u = 7 - x^3$; then $du/dx = -3x^2$. Hence

$$\begin{aligned}\int \frac{x^2}{(7-x^3)^7}\,dx &= -\tfrac{1}{3}\int \frac{1}{(7-x^3)^7}(-3x^2)\,dx\\ &= -\tfrac{1}{3}\int \frac{1}{u^7}\,du\\ &= -\tfrac{1}{3}\int u^{-7}\,du\\ &= -\tfrac{1}{3}\left(\frac{u^{-6}}{-6}\right) + c\\ &= \tfrac{1}{18}(7-x^3)^{-6} + c.\end{aligned}$$

(f) Here we use integration by parts.

Let $f(x) = x$ and $g'(x) = e^{-x/3}$; then

$$f'(x) = 1 \text{ and } g(x) = -3e^{-x/3}.$$

Hence

$$\begin{aligned}\int xe^{-x/3}\,dx &= x\left(-3e^{-x/3}\right) - \int 1\left(-3e^{-x/3}\right)dx\\ &= -3xe^{-x/3} + 3\int e^{-x/3}\,dx\\ &= -3xe^{-x/3} - 9e^{-x/3} + c.\end{aligned}$$

(g) Here

$$\int (x-1)^4\,dx = \frac{(x-1)^5}{5} + c.$$

Solution 4.1

The volume is

$$\pi \int_0^{\pi/4} \sec^2 x\, dx = \pi \left[\tan x\right]_0^{\pi/4}$$
$$= \pi(1-0)$$
$$= \pi$$
$$= 3.1416 \text{ (to 4 d.p.).}$$

Solution 4.2

The volume is

$$\pi \int_0^2 \left(\frac{\sqrt{x}}{2+x^2}\right)^2 dx = \pi \int_0^2 \frac{x}{(2+x^2)^2}\, dx.$$

Take $u = 2 + x^2$; then $du/dx = 2x$. Also $u = 2$ when $x = 0$, and $u = 6$ when $x = 2$. Hence

$$\pi \int_0^2 \frac{x}{(2+x^2)^2}\, dx = \frac{\pi}{2} \int_0^2 \frac{1}{(2+x^2)^2}(2x)\, dx$$
$$= \frac{\pi}{2} \int_2^6 u^{-2}\, du$$
$$= \frac{\pi}{2} \left[-u^{-1}\right]_2^6$$
$$= \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{6}\right)$$
$$= \frac{\pi}{6}$$
$$= 0.5236 \text{ (to 4 d.p.).}$$

Solution 4.3

The volume is

$$\pi \int_0^1 (xe^{4x})^2\, dx = \pi \int_0^1 x^2 e^{8x}\, dx.$$

Let $f(x) = x^2$ and $g'(x) = e^{8x}$; then

$$f'(x) = 2x \text{ and } g(x) = \tfrac{1}{8} e^{8x}.$$

Hence

$$\pi \int_0^1 x^2 e^{8x}\, dx = \pi \left[\tfrac{1}{8} x^2 e^{8x}\right]_0^1 - \frac{\pi}{4} \int_0^1 x e^{8x}\, dx.$$

Now use integration by parts again for the remaining integral.

Let $f(x) = x$ and $g'(x) = e^{8x}$; then

$$f'(x) = 1 \text{ and } g(x) = \tfrac{1}{8} e^{8x}.$$

Hence

$$\int_0^1 x e^{8x}\, dx = \left[\tfrac{1}{8} x e^{8x}\right]_0^1 - \tfrac{1}{8} \int_0^1 e^{8x}\, dx$$
$$= \left[\tfrac{1}{8} x e^{8x}\right]_0^1 - \left[\tfrac{1}{64} e^{8x}\right]_0^1$$
$$= \tfrac{1}{8} e^8 - \left(\tfrac{1}{64} e^8 - \tfrac{1}{64} e^0\right)$$
$$= \tfrac{7}{64} e^8 + \tfrac{1}{64}.$$

Thus the volume is

$$\pi \int_0^1 x^2 e^{8x}\, dx = \pi \left[\tfrac{1}{8} x^2 e^{8x}\right]_0^1 - \frac{\pi}{4} \int_0^1 x e^{8x}\, dx$$
$$= \pi \left(\left[\tfrac{1}{8} x^2 e^{8x}\right]_0^1 - \frac{1}{4}\left(\frac{7}{64} e^8 + \frac{1}{64}\right)\right)$$
$$= \pi \left(\frac{1}{8} e^8 - \frac{7}{256} e^8 - \frac{1}{256}\right)$$
$$= \frac{\pi}{256}\left(25 e^8 - 1\right)$$
$$= 914.53 \text{ (to 2 d.p.).}$$

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